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Stefan Tappe

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EXISTENCE OF AFFINE REALIZATIONS FOR LÉVY TERM STRUCTURE MODELS

STEFAN TAPPE

ABSTRACT. We investigate the existence of affine realizations for term structure models driven by Lévy processes. As it turns out, we obtain more severe restrictions on the volatility σ than in the classical diffusion case without jumps (see [8, 7, 23]). In particular, we show that for Lévy term structure models only two of the three well-known short rate models (Ho-Lee, Vasicek, CIR) still exist, namely the Ho-Lee and the Vasicek model.

Key Words: Geometry of Lévy term structure models, invariant foliations, affine realizations, short rate models.

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1. INTRODUCTION

In this paper, we investigate when a Lévy driven term structure model admits an affine realization. More precisely, we consider a market of zero coupon bonds

$$P(t,T) = \exp\left(- \int_t^{T-t} r_t(x)dx \right), \quad 0 \leq t \leq T \tag{1.1}$$

where the forward curves $x \mapsto r_t(x)$ (in the Musiela parametrization [33]) are given as the solution of a stochastic partial differential equation (SPDE) driven by a Lévy process X, the so-called HJMM (Heath–Jarrow–Morton–Musiela) equation

$$\begin{cases} dr_t &= (\frac{d}{dx} r_t + \alpha_{\mathrm{HJM}}(r_t))dt + \sigma(r_{t-})dX_t \\ r_0 &= h_0. \end{cases} \tag{1.2}$$

on a suitable Hilbert space H of forward curves, where $\frac{d}{dx}$ denotes the differential operator, which is generated by the strongly continuous semigroup $(S_t)_{t \geq 0}$ of shifts.

For classical Heath, Jarrow, Morton (HJM) models (see [26]) driven by a Wiener process, this question has been treated in [30, 36, 12, 1, 28, 3, 5, 10, 11] and finally completely been solved in [8, 7, 23], see also [2] for a survey.

Lévy term structure models, which generalize the classical HJM framework by incorporating jumps, have been proposed by Eberlein et al. [13, 14, 15, 16, 17, 18]. With a focus on SPDEs, they have been investigated in [21, 35, 32]. Other approaches in order to generalize the classical HJM framework can be found in Björk et al. [4, 6], Carmona and Tehranchi [9], and, e.g., [38, 29, 27].

The bond market (1.1) implied by a Lévy term structure model (1.2) is free of arbitrage if there exists an equivalent (local) martingale measure such that the discounted bond prices

$$\exp\left(- \int_0^t r_s(0)ds \right) P(t,T), \quad t \in [0,T]$$

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are local martingales for all maturities T. If we formulate the HJMM equation (1.2) with respect to such an equivalent martingale measure, then the drift is determined by the volatility, i.e. $\alpha_{\text{HJM}} : H \to H$ is given by the HJM drift condition

(1.3)
$$\alpha_{\text{HJM}}(h) := \frac{d}{dx}\Psi\bigg(- \int_0^\bullet \sigma(h)(\eta)d\eta \bigg) = -\sigma(h)\Psi'\bigg(- \int_0^\bullet \sigma(h)(\eta)d\eta \bigg), \quad h \in H$$

where Ψ denotes the cumulant generating function of the Lévy process, see [14, Sec. 2.1].

There are only very few references, such as [13, 31, 25, 27], that deal with affine realizations for arbitrage free term structure models with jumps.

In this text, we regard the problem, as in [8, 7, 23], from a geometric point of view, i.e. the forward rate process has to stay on a so-called foliation of affine manifolds (see Definition 2.6 below).

As our investigations will show, we obtain more severe restrictions on the volatility σ than in the classical diffusion case, see Theorem 8.1 and Corollary 8.3 below. In particular, we will show that for Lévy driven term structure models only two of the three well-known short rate models, which are

- the Ho-Lee model,
- the Hull-White extension of the Vasicek model,
- the Hull-White extension of the Cox-Ingersoll-Ross model

still exist, namely

- the Ho-Lee model,
- the Hull-White extension of the Vasicek model.

We will also have a closer look at the geometry of one of the remaining short rate models, namely the Vasicek model.

The remainder of this text is organized as follows: In Section 2 we provide results on invariant foliations and in Section 3 on affine realizations for general stochastic partial differential equations driven by Lévy processes. Afterwards, we introduce the space of forward curves in Section 4 and the term structure model in Section 5. After these preparations, we present a general result on affine realizations for Lévy term structure models in Section 6. Then, we study constant volatilities in Section 7, constant direction volatilities in Section 8, approximative realizations for constant direction volatilities in Section 9, and consequences for short rate realizations in Section 10.

2. Invariant foliations for stochastic partial differential equations driven by Lévy processes

In this section, we provide results on invariant foliations for general stochastic partial differential equations driven by Lévy processes, which we will apply to the HJMM equation (1.2) later on.

From now on, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let X be a real-valued Lévy process with Gaussian part $c \geq 0$ and Lévy measure F. In order to avoid trivialities, we assume that $c + F(\mathbb{R}) > 0$.

Here, we shall deal with stochastic partial differential equations of the type

(2.1)
$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_{t-})dX_t \\ r_0 &= h_0 \end{cases}$$

on a separable Hilbert space $(H, \| \cdot \|, \langle \cdot, \cdot \rangle)$. In (2.1), the operator $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $(S_t)_{t \geq 0}$ on H with adjoint operator $A^* : \mathcal{D}(A^*) \subset H \to H$. Recall that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense in H, see, e.g., [39, Satz VII.4.6, p. 351].

Concerning the vector fields $\alpha, \sigma : H \to H$ we impose the following condition.

2.1. **Assumption.** *We assume that* $\alpha, \sigma \in C^1(H)$ *and that there is a constant* $L > 0$ *such that*

$$\|\alpha(h_1) - \alpha(h_2)\| \leq L\|h_1 - h_2\|, \tag{2.2}$$

$$\|\sigma(h_1) - \sigma(h_2)\| \leq L\|h_1 - h_2\| \tag{2.3}$$

for all $h_1, h_2 \in H$.

The Lipschitz assumptions (2.2), (2.3) ensure that for each $h_0 \in H$ there exists a unique weak solution for (2.1) with $r_0 = h_0$.

2.2. **Definition.** *A subset* $U \subset H$ *is called* invariant *for (2.1) if for every* $h \in U$ *we have*

$$\mathbb{P}(r_t \in U) = 1 \quad \textit{for all } t \geq 0$$

where $(r_t)_{t \geq 0}$ *denotes the weak solution for (2.1) with* $r_0 = h$.

In what follows, let $V \subset H$ be a finite dimensional linear subspace and $d := \dim V$.

2.3. **Definition.** *A family* $(\mathcal{M}_t)_{t \geq 0}$ *of affine subspaces* $\mathcal{M}_t \subset H$, $t \geq 0$ *is called a* foliation generated by V *if there exists* $\psi \in C^1(\mathbb{R}_+; H)$ *such that*

$$\mathcal{M}_t = \psi(t) + V, \quad t \geq 0. \tag{2.4}$$

The map ψ *is a* parametrization *of the foliation* $(\mathcal{M}_t)_{t \geq 0}$.

2.4. **Remark.** *Note that the parametrization of a foliation* $(\mathcal{M}_t)_{t \geq 0}$ *generated by* V *is not unique. However, due to condition (2.4), for two parametrizations* ψ_1, ψ_2 *we have*

$$\psi_1(t) - \psi_2(t) \in V \quad \textit{for all } t \geq 0.$$

In what follows, let $(\mathcal{M}_t)_{t \geq 0}$ be a foliation generated by V. For every $t \geq 0$ the set $\pi_{V^\perp} \mathcal{M}_t$ consists of exactly one point. Therefore, the map

$$\psi : \mathbb{R}_+ \to H, \quad \psi(t) := \pi_{V^\perp} \mathcal{M}_t$$

is well-defined, and it is the unique parametrization of the foliation $(\mathcal{M}_t)_{t \geq 0}$ such that $\psi(t) \in V^\perp$ for all $t \geq 0$.

2.5. **Definition.** *For each* $t \geq 0$ *we define the* tangent space

$$T\mathcal{M}_t := \psi'(t) + V.$$

By Remark 2.4, the definition of the tangent is independent of the choice of the parametrization.

2.6. **Definition.** *The foliation* $(\mathcal{M}_t)_{t \geq 0}$ *of submanifolds is* invariant *for (2.1) if for every* $t_0 \in \mathbb{R}_+$ *and* $h \in \mathcal{M}_{t_0}$ *we have*

$$\mathbb{P}(r_t \in \mathcal{M}_{t_0 + t}) = 1 \quad \textit{for all } t \geq 0 \tag{2.5}$$

where $(r_t)_{t \geq 0}$ *denotes the weak solution for (2.1) with* $r_0 = h$.

As we shall see now, an invariant foliation generated by V, provided it exists, is unique.

2.7. **Lemma.** *Let* $(\mathcal{M}_t^i)_{t \geq 0}$, $i = 1, 2$ *be two foliations generated by* V *with* $\mathcal{M}_0^1 \cap \mathcal{M}_0^2 \neq \emptyset$, *which are invariant for (2.1). Then we have* $\mathcal{M}_t^1 = \mathcal{M}_t^2$ *for all* $t \geq 0$.

Proof. Choose $h_0 \in \mathcal{M}_0^1 \cap \mathcal{M}_0^2$ and let $(r_t)_{t\geq 0}$ be the weak solution for (2.1) with $r_0 = h_0$. Then we have

$$\pi_{V^\perp} \mathcal{M}_t^1 = \pi_{V^\perp} r_t = \pi_{V^\perp} \mathcal{M}_t^2, \quad t \geq 0$$

which completes the proof. □

2.8. **Proposition.** *Suppose the foliation $(\mathcal{M}_t)_{t\geq 0}$ of submanifolds is invariant for (2.1) and let $\ell \in L(H;\mathbb{R}^d)$ be a continuous linear operator with $\ell(V) = \mathbb{R}^d$. Then, for every $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ we have almost surely*

$$r_t = \pi_{V^\perp} \mathcal{M}_{t_0+t} + \ell^{-1}(\ell(r_t) - \ell \pi_{V^\perp} \mathcal{M}_{t_0+t}), \quad t \geq 0 \tag{2.6}$$

where $(r_t)_{t\geq 0}$ denotes the weak solution for (2.1) with $r_0 = h$, and (2.6) is the decomposition of $(r_t)_{t\geq 0}$ according to $V^\perp \oplus V$.

Proof. By condition (2.5) we obtain almost surely

$$r_t = \pi_{V^\perp} r_t + \pi_V r_t = \pi_{V^\perp} \mathcal{M}_{t_0+t} + \pi_V r_t, \quad t \geq 0. \tag{2.7}$$

Therefore we obtain almost surely

$$\pi_V r_t = r_t - \pi_{V^\perp} \mathcal{M}_{t_0+t} = \ell^{-1}(\ell(r_t) - \ell(\pi_{V^\perp} \mathcal{M}_{t_0+t})), \quad t \geq 0. \tag{2.8}$$

Inserting (2.8) into (2.7), we arrive at (2.6). □

2.9. **Remark.** *If the foliation $(\mathcal{M}_t)_{t\geq 0}$ is invariant for (2.1), then for every continuous linear operator $\ell \in L(H;\mathbb{R}^d)$ with $\ell(V) = \mathbb{R}^d$ the decomposition (2.6) provides a realization of the solution $(r_t)_{t\geq 0}$ by means of the finite dimensional process $\ell(r)$.*

We shall now approach our main result of this section, Theorem 2.11 below, which provides consistency conditions for invariance of the foliation $(\mathcal{M}_t)_{t\geq 0}$.

2.10. **Lemma.** *There exist $\zeta_1, \ldots, \zeta_d \in \mathcal{D}(A^*)$ and an isomorphism $\phi : \mathbb{R}^d \to V$ such that*

$$\phi(\langle \zeta, h \rangle) = h \quad \textit{for all } h \in V, \tag{2.9}$$

where we use the notation $\langle \zeta, h \rangle := (\langle \zeta_1, h \rangle, \ldots, \langle \zeta_d, h \rangle) \in \mathbb{R}^d$.

Proof. By the Gram-Schmidt method, there exists an orthonormal basis $\{e_1, \ldots, e_d\}$ of V. Since $\mathcal{D}(A^*)$ is dense in H, there exist $\zeta_1, \ldots, \zeta_d \in \mathcal{D}(A^*)$ with $\|\zeta_i - e_i\| < 2^{-d}$ for $i = 1, \ldots, d$. Hence, we obtain

$$|\langle \zeta_i, e_j \rangle| \leq |\langle e_i, e_j \rangle| + |\langle \zeta_i - e_i, e_j \rangle| < 2^{-d}$$

for all $i, j = 1, \ldots, d$ with $i \neq j$ and

$$|\langle \zeta_i, e_i \rangle| \geq |\langle e_i, e_i \rangle| - |\langle \zeta_i - e_i, e_i \rangle| > 1 - 2^{-d}$$

for all $i = 1, \ldots, d$. Thus, we have

$$\sum_{\substack{j=1 \\ j \neq i}}^{d} |\langle \zeta_i, e_j \rangle| < (d-1)2^{-d} < (2^d - 1)2^{-d} = 1 - 2^d < |\langle \zeta_i, e_i \rangle|$$

for all $i = 1, \ldots, d$, and hence, due to the Theorem of Gerschgorin, the $(d \times d)$-matrix

$$B := \begin{pmatrix} \langle \zeta_1, e_1 \rangle & \cdots & \langle \zeta_1, e_d \rangle \\ \vdots & & \vdots \\ \langle \zeta_d, e_1 \rangle & \cdots & \langle \zeta_d, e_d \rangle \end{pmatrix}$$

is invertible. Let $\psi : V \to \mathbb{R}^d$ be the isomorphism

$$\psi(h) := (\langle e_1, h \rangle, \ldots, \langle e_d, h \rangle), \quad h \in V.$$

Then, the isomorphism $B\psi : V \to \mathbb{R}^d$ has the representation

$$B\psi(h) = (\langle \zeta_1, h \rangle, \dots, \langle \zeta_d, h \rangle), \quad h \in V.$$

Defining the isomorphism $\phi := (B\psi)^{-1} : \mathbb{R}^d \to V$ completes the proof. □

Now, let $\psi \in C^1(\mathbb{R}_+; H)$ be a parametrization of $(\mathcal{M}_t)_{t \geq 0}$ and let $\phi : \mathbb{R}^d \to V$ be an isomorphism as in Lemma 2.10. We define $\tilde{\alpha}, \tilde{\sigma} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ as

$$\begin{aligned} \tilde{\alpha}(t, z) &:= \langle A^* \zeta, \psi(t) + \phi(z) \rangle + \langle \zeta, \alpha(\psi(t) + \phi(z)) - \psi'(t) \rangle, \\ \tilde{\sigma}(t, z) &:= \langle \zeta, \sigma(\psi(t) + \phi(z)) \rangle. \end{aligned}$$

By Assumption 2.1 we have $\tilde{\alpha}, \tilde{\sigma} \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ and there exists a constant $K > 0$ such that

$$\begin{aligned} \|\tilde{\alpha}(t, z_1) - \tilde{\alpha}(t, z_2)\|_{\mathbb{R}^d} &\leq K \|z_1 - z_2\|_{\mathbb{R}^d} \\ \|\tilde{\sigma}(t, z_1) - \tilde{\sigma}(t, z_2)\|_{\mathbb{R}^d} &\leq K \|z_1 - z_2\|_{\mathbb{R}^d} \end{aligned}$$

for all $t \in \mathbb{R}_+$ and all $z_1, z_2 \in \mathbb{R}^d$. Thus, for each $t_0 \in \mathbb{R}_+$ and each $z_0 \in \mathbb{R}^d$ there exists a unique strong solution for

$$\left\{ \begin{array}{rcl} dZ_t &=& \tilde{\alpha}(t_0 + t, Z_t)dt + \tilde{\sigma}(t_0 + t, Z_{t-})dX_t \\ Z_0 &=& z_0. \end{array} \right. \tag{2.10}$$

We define the vector field $\nu : \mathcal{D}(A) \to H$ as

$$\nu(h) := Ah + \alpha(h), \quad h \in \mathcal{D}(A).$$

Here is our main result concerning invariance of the foliation $(\mathcal{M}_t)_{t \geq 0}$ for the Lévy driven SPDE (2.1).

2.11. **Theorem.** *The foliation $(\mathcal{M}_t)_{t \geq 0}$ is an invariant foliation for (2.1) if and only if for all $t \geq 0$ we have*

$$\mathcal{M}_t \subset \mathcal{D}(A), \tag{2.11}$$

$$\nu(h) \in T\mathcal{M}_t, \quad h \in \mathcal{M}_t \tag{2.12}$$

$$\sigma(h) \in V, \quad h \in \mathcal{M}_t. \tag{2.13}$$

If the previous conditions are satisfied, the map

$$\mathbb{R}_+ \to H, \quad t \mapsto A(\pi_{V^\perp} \mathcal{M}_t) \tag{2.14}$$

is continuous, and for every $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ the weak solution for (2.1) with $r_0 = h$ is also a strong solution.

Proof. "⇒": Let $t_0 \in \mathbb{R}_+$ and $h_0 \in V$ be arbitrary. Then we have $h := \psi(t_0) + h_0 \in \mathcal{M}_{t_0}$. Let $(r_t)_{t \geq 0}$ be the weak solution for (2.1) with $r_0 = h$ and set $z_0 := \langle \zeta, h_0 \rangle$. Since $\zeta_1, \dots, \zeta_d \in \mathcal{D}(A^*)$ and $(\mathcal{M}_t)_{t \geq 0}$ is an invariant foliation for (2.1), we obtain, by using (2.9),

$$\begin{aligned} \langle \zeta, r_t - \psi(t_0 + t) \rangle &= \langle \zeta, h - \psi(t_0) \rangle + \int_0^t (\langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) - \psi'(t_0 + s) \rangle) ds \\ &\quad + \int_0^t \langle \zeta, \sigma(r_{s-}) \rangle dX_s \\ &= \langle \zeta, h_0 \rangle + \int_0^t \tilde{\alpha}(t_0 + s, \langle \zeta, r_s - \psi(t_0 + s) \rangle) ds \\ &\quad + \int_0^t \tilde{\sigma}(t_0 + s, \langle \zeta, r_{s-} - \psi(t_0 + s) \rangle) dX_s. \end{aligned}$$

This identity shows that almost surely

$$Z_t = \langle \zeta, r_t - \psi(t_0 + t) \rangle, \quad t \geq 0$$

where $(Z_t)_{t \geq 0}$ denotes the strong solution for (2.10) with $Z_0 = z_0$. By (2.9), we have almost surely

$$\phi(Z_t) = r_t - \psi(t_0 + t), \quad t \geq 0.$$

Let $\xi \in \mathcal{D}(A^*)$ be arbitrary. We obtain, by Itô's formula and applying the linear functional $\langle \xi, \cdot \rangle$ afterwards,

(2.15)
$$\langle \xi, r_t - \psi(t_0 + t) \rangle = \langle \xi, \phi(Z_t) \rangle = \langle \xi, h_0 - \psi(t_0) \rangle + \int_0^t \langle \xi, \phi(\tilde{\alpha}(t_0 + s, Z_s)) \rangle ds$$
$$+ \int_0^t \langle \xi, \phi(\tilde{\sigma}(t_0 + s, Z_{s-})) \rangle dX_s.$$

Since $(r_t)_{t \geq 0}$ is a weak solution for (2.1) with $r_0 = h_0$, we have

(2.16)
$$\langle \xi, r_t - \psi(t_0 + t) \rangle = \langle \xi, h_0 - \psi(t_0) \rangle + \int_0^t (\langle A^* \xi, r_s \rangle + \langle \xi, \alpha(r_s) - \psi'(t_0 + s) \rangle) ds$$
$$+ \int_0^t \langle \xi, \sigma(r_{s-}) \rangle dX_s.$$

Combining (2.15) and (2.16) we get

$$\begin{aligned} 0 = & \int_0^t (\langle A^* \xi, r_s \rangle + \langle \xi, \alpha(r_s) - \psi'(t_0 + s) - \phi(\tilde{\alpha}(t_0 + s, Z_s)) \rangle) ds \\ & + \int_0^t \langle \xi, \sigma(r_{s-}) - \phi(\tilde{\sigma}(t_0 + s, Z_{s-})) \rangle dX_s. \end{aligned} \tag{2.17}$$

Therefore, all integrands in (2.17) vanish and, since $\xi \in \mathcal{D}(A^*)$ was arbitrary, setting $s = 0$ yields $\psi(t_0) + h_0 \in \mathcal{D}(A)$, proving (2.11) and the identities

$$\nu(\psi(t_0) + h_0) = \psi'(t_0) + \phi(\tilde{\alpha}(t_0, z_0)) \in T\mathcal{M}_{t_0}, \tag{2.18}$$

$$\sigma(\psi(t_0) + h_0) = \phi(\tilde{\sigma}(t_0, z_0)) \in V \tag{2.19}$$

which show (2.12), (2.13). Furthermore, identity (2.18) proves the continuity of the map defined in (2.14).

"$\Leftarrow$": Let $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ be arbitrary. There exists a unique $z_0 \in \mathbb{R}^d$ such that $h = \psi(t_0) + \phi(z_0)$. Let $(Z_t)_{t \geq 0}$ be the strong solution for (2.10) with $Z_0 = z_0$. Itô's formula yields, by using (2.11)–(2.13) and (2.9),

$$\begin{aligned} \psi(t_0 + t) + \phi(Z_t) &= \psi(t_0) + \phi(z_0) + \int_0^t (\psi'(t_0 + s) + \phi(\tilde{\alpha}(t_0 + s, Z_s))) ds \\ &\quad + \int_0^t \phi(\tilde{\sigma}(t_0 + s, Z_{s-})) dX_s \\ &= h + \int_0^t (\psi'(t_0 + s) + \phi(\langle \zeta, \nu(\psi(t_0 + s) + \phi(Z_s)) - \psi'(t_0 + s) \rangle)) ds \\ &\quad + \int_0^t \phi(\langle \zeta, \sigma(\psi(t_0 + s) + \phi(Z_{s-})) \rangle) dX_s \\ &= h + \int_0^t (A(\psi(t_0 + s) + \phi(Z_s)) + \alpha(\psi(t_0 + s) + \phi(Z_s))) ds \\ &\quad + \int_0^t \sigma(\psi(t_0 + s) + \phi(Z_{s-})) dX_s, \quad t \geq 0. \end{aligned}$$

By the uniqueness of solutions for (2.1) we obtain almost surely

$$r_t = \psi(t_0 + t) + \phi(Z_t) \in \mathcal{M}_{t_0+t}, \quad t \geq 0$$

where $(r_t)_{t\geq 0}$ denotes the weak solution for (2.1) with $r_0 = h$, whence $(\mathcal{M}_t)_{t\geq 0}$ is an invariant foliation, and we get that $(r_t)_{t\geq 0}$ is also a strong solution. □

2.12. **Remark.** *Note that (2.11)–(2.13) are consistency conditions on the tangent spaces (for related results see, e.g., [19]). Since the foliation $(\mathcal{M}_t)_{t\geq 0}$ consists of affine manifolds, the jumps of the Lévy process X do not matter, and we do not need a Stratonovich correction term for the drift.*

Now, we express the consistency conditions from Theorem 2.11 by means of a coordinate system. Let $\psi \in C^1(\mathbb{R}_+; H)$ be a parametrization of $(\mathcal{M}_t)_{t\geq 0}$ and let $\{\lambda_1, \dots, \lambda_d\}$ be a basis of V.

2.13. **Corollary.** *The following statements are equivalent:*

(1) $(\mathcal{M}_t)_{t\geq 0}$ *is an invariant foliation for (2.1).*
(2) *We have*

$$\psi(\mathbb{R}_+) \subset \mathcal{D}(A), \tag{2.20}$$

$$\lambda_1, \dots, \lambda_d \in \mathcal{D}(A) \tag{2.21}$$

and there exist $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ *such that*

$$\nu\Big(\psi(t) + \sum_{i=1}^d y_i \lambda_i\Big) = \psi'(t) + \sum_{i=1}^d \mu_i(t,y)\lambda_i, \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}^d \tag{2.22}$$

$$\sigma\Big(\psi(t) + \sum_{i=1}^d y_i \lambda_i\Big) = \sum_{i=1}^d \gamma_i(t,y)\lambda_i, \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{2.23}$$

If the previous conditions are satisfied, μ and γ are uniquely determined, we have $\mu, \gamma \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$, there exists a constant $K > 0$ such that

$$\|\mu(t,y_1) - \mu(t,y_2)\|_{\mathbb{R}^d} \leq K\|y_1 - y_2\|_{\mathbb{R}^d} \tag{2.24}$$

$$\|\gamma(t,y_1) - \gamma(t,y_2)\|_{\mathbb{R}^d} \leq K\|y_1 - y_2\|_{\mathbb{R}^d} \tag{2.25}$$

for all $t \in \mathbb{R}_+$ and $y_1, y_2 \in \mathbb{R}^d$, and for every $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ the weak solution for (2.1) with $r_0 = h$ is also a strong solution.

Proof. The asserted equivalence follows from Theorem 2.11. By the linear independence of $\lambda_1, \dots, \lambda_d$, the mappings μ and γ are uniquely determined. Denoting by $\pi : \mathbb{R}^d \to V$ the isomorphism $\pi(y) := \sum_{i=1}^d y_i \lambda_i$, we can express them as

$$\mu(t,y) = \pi^{-1}\bigg(\nu\Big(\psi(t) + \sum_{i=1}^d y_i \lambda_i\Big) - \psi'(t)\bigg),$$

$$\gamma(t,y) = \pi^{-1}\bigg(\sigma\Big(\psi(t) + \sum_{i=1}^d y_i \lambda_i\Big)\bigg).$$

Since the map defined in (2.14) is continuous by Theorem 2.11, we have $\mu, \gamma \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ and (2.24), (2.25) by virtue of Assumption 2.1. □

Suppose the foliation $(\mathcal{M}_t)_{t\geq 0}$ is invariant for (2.1). We shall now identify the underlying coordinate process Y. Let $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ be arbitrary. There exists a unique $y \in \mathbb{R}^d$ such that $h = \psi(t_0) + \sum_{i=1}^d y_i \lambda_i$. Taking into account (2.24), (2.25), we let $(Y_t)_{t \geq 0}$ be the strong solution for

$$\left\{ \begin{array}{rcl} dY_t & = & \mu(t_0 + t, Y_t)dt + \gamma(t_0 + t, Y_{t-})dX_t \\ Y_0 & = & y, \end{array} \right. \tag{2.26}$$

where $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are given by (2.22), (2.23). By Itô's formula, the process

$$r_t = \psi(t_0 + t) + \sum_{i=1}^d Y_t^i \lambda_i, \quad t \geq 0 \tag{2.27}$$

is the strong solution for (2.1) with $r_0 = h$.

2.14. **Remark.** *If we think of interest rate models, the state process Y has no direct economic interpretation. Proposition 2.8 shows that for any continuous linear operator $\ell : H \to \mathbb{R}^d$ with $\ell(V) = \mathbb{R}^d$ we can choose $\ell(r)$ as state process. We may think of $\ell_i(h) = \frac{1}{x_i} \int_0^{x_i} h(\eta) d\eta$ (benchmark yields) or $\ell_i(h) = h(x_i)$ (benchmark forward rates). We refer to* [7, Sec. 7], [5, Prop. 5.1], [8, Thm. 3.3], [11, Prop. 2], [12, Sec. 5] *for related results.*

3. Affine realizations for stochastic partial differential equations driven by Lévy processes

The results of the previous section lead to the following definition of an affine realization.

3.1. **Definition.** *Let $V \subset H$ be a finite dimensional linear subspace. The SPDE (2.1) has an* affine realization generated by V *if for each $h_0 \in \mathcal{D}(A)$ there exists a foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ generated by V with $h_0 \in \mathcal{M}_0^{h_0}$, which is invariant for (2.1).*

We call $d := \dim V$ the *dimension* of the affine realization.

3.2. **Lemma.** *Let $d \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_d \in H$ be linearly independent. Suppose the SPDE (2.1) has a d-dimensional affine realization generated by $V = \langle \lambda_1, \ldots, \lambda_d \rangle$. Then, there exist $\Phi_1, \ldots, \Phi_d \in C^1(H; \mathbb{R})$ such that*

$$\sigma(h) = \sum_{i=1}^d \Phi_i(h) \lambda_i, \quad h \in H. \tag{3.1}$$

Proof. Relation (2.13) from Theorem 2.11 yields $\sigma(h) \in V$ for all $h \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in H and V is closed, we obtain $\sigma(h) \in V$ for all $h \in H$. Hence, there exist $\Phi_1, \ldots, \Phi_d : H \to \mathbb{R}$ such that (3.1) is satisfied. Since $\sigma \in C^1(H)$, we have $\Phi_1, \ldots, \Phi_d \in C^1(H; \mathbb{R})$. □

Suppose the SPDE (2.1) has an affine realization generated by a finite dimensional subspace $V \subset H$. Then, for each $h_0 \in \mathcal{D}(A)$ the foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ is uniquely determined by Lemma 2.7. We define the *singular set* Σ as

$$\begin{aligned} \Sigma &= \{h_0 \in \mathcal{D}(A) : \mathcal{M}_0^{h_0} = \mathcal{M}_t^{h_0} \text{ for all } t \geq 0\} \\ &= \{h_0 \in \mathcal{D}(A) : h_0 + V \text{ is an invariant manifold}\}. \end{aligned}$$

A consequence of this definition is the identity

$$\Sigma + V = \Sigma. \tag{3.2}$$

In particular, Σ is an invariant set for (2.1).

3.3. **Proposition.** *Suppose the SPDE (2.1) has an affine realization generated by V. Then, the singular set Σ is given by*

$$\Sigma = \{h_0 \in \mathcal{D}(A) : \nu(h_0) \in V\}, \tag{3.3}$$

for each $h_0 \in \mathcal{D}(A)$ the weak solution $(r_t)_{t\geq 0}$ for (2.1) with $r_0 = h_0$ is also a strong solution, and we have

$$\mathbb{P}(r_t \notin \Sigma) = 1, \quad t \in [0, t_0) \tag{3.4}$$

$$\mathbb{P}(r_t \in \Sigma) = 1, \quad t \in [t_0, \infty) \tag{3.5}$$

where we have set

$$t_0 := \inf\{t \geq 0 : \mathcal{M}_t^{h_0} \subset \Sigma\} \in [0, \infty].$$

Proof. Let $h_0 \in \mathcal{D}(A)$ be arbitrary. By condition (2.12) of Theorem 2.11 we have $\nu(h_0) \in V$ if and only if $\nu(h) \in V$ for all $h \in h_0 + V$, which means that $h_0 + V$ is an invariant manifold, proving (3.3). By Theorem 2.11, the weak solution $(r_t)_{t\geq 0}$ for (2.1) with $r_0 = h_0$ is also a strong solution, and, by taking into account (3.2), we obtain (3.4) and (3.5). □

3.4. **Remark.** *Note that the time t_0 is deterministic. By (3.2), for any parametrization ψ of the foliation $(\mathcal{M}_t^{h_0})_{t\geq 0}$ it is given by*

$$t_0 = \inf\{t \geq 0 : \psi(t) \in \Sigma\}.$$

We observe the following dichotomic behaviour of the solutions for (2.1). Up to time t_0, the solution proceeds outside the singular Σ, afterwards it stays in Σ, and therefore even on an affine manifold. In particular, if $t_0 = 0$ we have $\mathbb{P}(r_t \in \Sigma) = 1$ for all $t \geq 0$, and if $t_0 = \infty$ we have $\mathbb{P}(r_t \notin \Sigma) = 1$ for all $t \geq 0$.

For our later investigations on the existence of affine realizations, quasi-exponential functions (cf. [8, Sec. 5]), which we shall now introduce in this general context, will play an important role. Inductively, we define the domains

$$\mathcal{D}(A^n) := \{h \in \mathcal{D}(A^{n-1}) : A^{n-1}h \in \mathcal{D}(A)\}, \quad n \geq 2$$

as well as the intersection

$$\mathcal{D}(A^\infty) := \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n).$$

3.5. **Definition.** *An element $h \in \mathcal{D}(A^\infty)$ is called* quasi-exponential *if*

$$\dim\langle A^n h : n \in \mathbb{N}_0\rangle < \infty. \tag{3.6}$$

3.6. **Lemma.** *Let $h \in H$, $h \neq 0$ be arbitrary. The following statements are equivalent.*

1. *h is quasi-exponential.*
2. *We have $h \in \mathcal{D}(A^\infty)$ and there exists $d \in \mathbb{N}$ such that $\{h, Ah, \dots, A^{d-1}h\}$ is a basis of $\langle A^n h : n \in \mathbb{N}_0\rangle$.*
3. *There exists a finite dimensional subspace $V \subset \mathcal{D}(A)$ with $h \in V$ such that*

$$Av \in V \quad \text{for all } v \in V. \tag{3.7}$$

Proof. (1) ⇒ (2): Since $h \neq 0$ is quasi-exponential, there exists a minimal integer $d \in \mathbb{N}$ such that $h, Ah, \dots, A^{d-1}h$ are linearly independent. By induction, we show that

$$A^n h \in \langle h, Ah, \dots, A^{d-1}h\rangle \quad \text{for all } n \geq d,$$

whence $\{h, Ah, \dots, A^{d-1}h\}$ is a basis of $\langle A^n h : n \in \mathbb{N}_0\rangle$.

(2) ⇒ (3): The finite dimensional subspace $V = \langle h, Ah, \dots, A^{d-1}h\rangle$ has the desired properties.

(3) $\Rightarrow$ (1): Using (3.7), by induction, for each $n \in \mathbb{N}$ we have $h \in \mathcal{D}(A^n)$ and $A^n h \in V$, which yields $h \in \mathcal{D}(A^\infty)$ and (3.6), whence h is quasi-exponential. $\square$

4. The space of forward curves

In this section, we define the space of forward curves, on which we will study the HJMM equation (1.2) in the forthcoming sections. These spaces have been introduced in [20, Sec. 5].

We fix an arbitrary constant $\beta > 0$. Let H_β be the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \right)^{\frac{1}{2}} < \infty. \tag{4.1}$$

Let $(S_t)_{t \geq 0}$ be the shift semigroup on H_β defined by $S_t h := h(t + \cdot)$ for $t \in \mathbb{R}_+$.

Since forward curves should flatten for large time to maturity x, the choice of H_β is reasonable from an economic point of view.

4.1. **Theorem.** *Let $\beta > 0$ be arbitrary.*

1. *The space $(H_\beta, \| \cdot \|_\beta)$ is a separable Hilbert space.*
2. *For each $x \in \mathbb{R}_+$, the point evaluation $h \mapsto h(x) : H_\beta \to \mathbb{R}$ is a continuous linear functional.*
3. *$(S_t)_{t \geq 0}$ is a C_0-semigroup on H_β with infinitesimal generator $\frac{d}{dx} : \mathcal{D}(\frac{d}{dx}) \subset H_\beta \to H_\beta$, $\frac{d}{dx} h = h'$, and domain*
$$\mathcal{D}\left(\frac{d}{dx} \right) = \{ h \in H_\beta : h' \in H_\beta \}.$$
4. *Each $h \in H_\beta$ is continuous, bounded and the limit $h(\infty) := \lim_{x \to \infty} h(x)$ exists.*
5. *$H_\beta^0 := \{ h \in H_\beta : h(\infty) = 0 \}$ is a closed subspace of H_β.*
6. *There exists a universal constant $C > 0$, only depending on β, such that for all $h \in H_\beta$ we have the estimate*
$$\|h\|_{L^\infty(\mathbb{R}_+)} \leq C \|h\|_\beta, \tag{4.2}$$
7. *For each $\beta' > \beta$, we have $H_{\beta'} \subset H_\beta$ and the relation*
$$\|h\|_\beta \leq \|h\|_{\beta'}, \quad h \in H_{\beta'}. \tag{4.3}$$

Proof. Note that H_β is the space H_w from [20, Sec. 5.1] with weight function $w(x) = e^{\beta x}$, $x \in \mathbb{R}_+$. Hence, the first six statements follow from [20, Thm. 5.1.1, Cor. 5.1.1]. For each $\beta' > \beta$, the observation

$$\int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \leq \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta' x} dx, \quad h \in H_{\beta'}$$

shows $H_{\beta'} \subset H_\beta$ and (4.3). $\square$

4.2. **Lemma.** *The following statements are valid.*

1. *For all $h, g \in H_\beta$ we have $hg \in H_\beta$ and the multiplication map $m : H_\beta \times H_\beta \to H_\beta$ defined as $m(h,g) := hg$ is a continuous, bilinear operator.*
2. *For all $h, g \in \mathcal{D}(\frac{d}{dx})$ we have $hg \in \mathcal{D}(\frac{d}{dx})$.*

Proof. The function hg is absolutely continuous, because h and g are absolutely continuous and bounded, see Theorem 4.1. By estimate (4.2) we obtain

$$\begin{aligned} \|hg\|_\beta^2 &= |h(0)|^2|g(0)|^2 + \int_{\mathbb{R}_+} |h(x)g'(x) + g(x)h'(x)|^2 e^{\beta x} dx \\ &\leq \|h\|_{L^\infty(\mathbb{R}_+)}^2 \|g\|_{L^\infty(\mathbb{R}_+)}^2 + 2\|h\|_{L^\infty(\mathbb{R}_+)}^2 \int_{\mathbb{R}_+} |g'(x)|^2 e^{\beta x} dx \\ &\quad + 2\|g\|_{L^\infty(\mathbb{R}_+)}^2 \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \\ &\leq C^4 \|h\|_\beta^2 \|g\|_\beta^2 + 2C^2 \|h\|_\beta^2 \|g\|_\beta^2 + 2C^2 \|g\|_\beta^2 \|h\|_\beta^2 < \infty. \end{aligned}$$

Hence, we have $hg \in H_\beta$ and the estimate

$$\|m(h,g)\|_\beta \leq \sqrt{C^4 + 4C^2} \|h\|_\beta \|g\|_\beta,$$

proving that m is a continuous, bilinear operator.

If $h, g \in \mathcal{D}(\frac{d}{dx})$, we have $hg \in C^1(\mathbb{R}_+)$ with $(hg)' = h'g + hg'$, whence $hg \in \mathcal{D}(\frac{d}{dx})$ by the first statement. □

For $\lambda \in H_\beta$ we define $\Lambda := \mathcal{I}\lambda := \int_0^\bullet \lambda(\eta) d\eta$, which belongs to $C(\mathbb{R}_+)$, the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$.

4.3. **Lemma.** *Let $\lambda \in H_\beta$ with $\lambda \neq 0$ be arbitrary. Then, the families $\{\Lambda^n : n \in \mathbb{N}_0\}$ and $\{\lambda\Lambda^n : n \in \mathbb{N}_0\}$ are linearly independent in $C(\mathbb{R}_+)$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. Since $\lambda \neq 0$, there exist $x_1, \ldots, x_n \in \mathbb{R}_+$ such that $\lambda(x_i) \neq 0$, $i = 1, \ldots, n$ and $\Lambda(x_1), \ldots, \Lambda(x_n)$ are pairwise different. Therefore, the Vandermonde matrix

$$\begin{pmatrix} 1 & \Lambda(x_1) & \Lambda^2(x_1) & \cdots & \Lambda^{n-1}(x_1) \\ 1 & \Lambda(x_2) & \Lambda^2(x_2) & \cdots & \Lambda^{n-1}(x_2) \\ \vdots & & \vdots & \ddots & \vdots \\ 1 & \Lambda(x_n) & \Lambda^2(x_n) & \cdots & \Lambda^{n-1}(x_n) \end{pmatrix}$$

is invertible, and hence the families $\{\Lambda^n : n \in \mathbb{N}_0\}$ and $\{\lambda\Lambda^n : n \in \mathbb{N}_0\}$ are linearly independent in $C(\mathbb{R}_+)$. □

4.4. **Lemma.** *Let $0 < \beta < \beta'$ be arbitrary real numbers. For each $\lambda \in H_{\beta'}^0$ we have $\Lambda \in H_\beta$ and the map $\mathcal{I} : H_{\beta'}^0 \to H_\beta$ is a continuous linear operator.*

Proof. Let $\lambda \in H_{\beta'}^0$ be arbitrary. Then $\mathcal{I}\lambda$ is absolutely continuous. Since $\mathcal{I}\lambda(0) = 0$, using Hölder's inequality, we obtain

$$\begin{aligned} \|\mathcal{I}\lambda\|_\beta^2 &= \int_{\mathbb{R}_+} \lambda(x)^2 e^{\beta x} dx = \int_{\mathbb{R}_+} \left(\int_x^\infty \lambda'(y) e^{\frac{1}{2}\beta' y} e^{-\frac{1}{2}\beta' y} dy \right)^2 e^{\beta x} dx \\ &\leq \int_{\mathbb{R}_+} \left(\int_x^\infty \lambda'(y)^2 e^{\beta' y} dy \right) \left(\int_x^\infty e^{-\beta' y} dy \right) e^{\beta x} dx \\ &\leq \|\lambda\|_{\beta'}^2 \int_{\mathbb{R}_+} \frac{1}{\beta'} e^{-(\beta' - \beta)x} dx = \frac{1}{\beta'(\beta' - \beta)} \|\lambda\|_{\beta'}^2, \end{aligned}$$

proving the assertion. □

5. Presentation of the term structure model

We shall now introduce the Lévy term structure model by using the space of forward curves from the previous section.

Recall that $c \geq 0$ denotes the Gaussian part and F the Lévy measure of the Lévy process X.

5.1. **Assumption.** *We assume there exist constants $N, \epsilon > 0$ such that*

$$\int_{\{|x|>1\}} e^{zx} F(dx) < \infty, \quad z \in [-(1+\epsilon)N, (1+\epsilon)N].$$

Then, the cumulant generating function

$$\Psi(z) := \ln \mathbb{E}[e^{zX_1}]$$

exists on $[-N, N]$ and is of class C^∞, see [37, Lemma 26.4]. In particular, there exists a constant $K > 0$ such that for all $k = 1, \ldots, e$ we have

$$|\Psi_k^{(n)}(x)| \leq K, \quad x \in [-N, N] \text{ and } n = 0, \ldots, 4. \tag{5.1}$$

We can write the cumulant generating function as

$$\Psi(z) = bz + \frac{c}{2} z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx) F(dx), \quad z \in [-N, N]$$

where $b \in \mathbb{R}$ denotes the drift of X. Differentiating and evaluating at zero, we see that

$$\Psi''(0) = c + \int_{\mathbb{R}} x^2 F(dx), \tag{5.2}$$

$$\Psi^{(n)}(0) = \int_{\mathbb{R}} x^n F(dx) \quad \text{for } n \geq 3. \tag{5.3}$$

Let $0 < \beta < \beta'$ be arbitrary real numbers and let a volatility structure $\sigma : H_\beta \to H_\beta$ be given. We define the open set $\mathcal{O} \subset H_\beta$ as

$$\mathcal{O} := \{h \in H_\beta : \|h\|_\beta < \tfrac{N}{C}\},$$

By estimate (4.2) we have $\mathcal{O} \subset U$, where $U \subset H_\beta$ denotes the subset

$$U := \{h \in H_\beta : \|h\|_{L^\infty(\mathbb{R}_+)} \leq N\}.$$

5.2. **Assumption.** *We assume that $\sigma \in C^1(H_\beta)$ with $\sigma(H_\beta) \subset H^0_{\beta'}$, $-\mathcal{I}\sigma(H_\beta) \subset \mathcal{O}$, and that there exist $L, M > 0$ such that*

$$\|\sigma(h_1) - \sigma(h_2)\|_\beta \leq L\|h_1 - h_2\|_\beta \quad \textit{for all } h_1, h_2 \in H_\beta,$$
$$\|\sigma(h)\|_\beta \leq M \quad \textit{for all } h \in H_\beta.$$

According to the HJM drift condition (1.3) we define

$$\alpha_{\mathrm{HJM}}(h) := \frac{d}{dx} \Psi\Big(-\int_0^\bullet \sigma(h)(\eta) d\eta \Big) = -\sigma(h) \Psi'\Big(-\int_0^\bullet \sigma(h)(\eta) d\eta \Big), \quad h \in H_\beta. \tag{5.4}$$

By [21, Prop. 4.5] we have $\alpha_{\mathrm{HJM}}(H_\beta) \subset H^0_\beta$ and there exists a constant $\tilde{L} > 0$ such that

$$\|\alpha_{\mathrm{HJM}}(h_1) - \alpha_{\mathrm{HJM}}(h_2)\|_\beta \leq \tilde{L}\|h_1 - h_2\|_\beta \quad \text{for all } h_1, h_2 \in H_\beta. \tag{5.5}$$

Hence, for each $h_0 \in H_\beta$ there exists a unique weak solution for (1.2) with $r_0 = h_0$. Note that (1.2) is a particular example of the stochastic partial differential equation (2.1) with $A = \frac{d}{dx}$ and $\alpha = \alpha_{\mathrm{HJM}}$.

In order to show smoothness of the drift term α_{HJM}, we provide the following auxiliary result.

5.3. **Lemma.** *For each $h \in U$ we have $\Psi'(h) \in H_\beta$ and the map $\Psi' : \mathcal{O} \to H_\beta$ is Fréchet differentiable with derivative*

$$D\Psi'(h) \bullet v = \Psi''(h) v.$$

Proof. The map $\Psi'(h)$ is again absolutely continuous because of (5.1). Moreover, by (5.1), we have

$$\begin{aligned}\|\Psi'(h)\|_\beta^2 &= |\Psi'(h(0))|^2 + \int_{\mathbb{R}_+} |\Psi''(h(x))h'(x)|^2 e^{\beta x} dx \\ &\leq |\Psi'(h(0))|^2 + K^2 \|h\|_\beta^2 < \infty,\end{aligned}$$

whence $\Psi'(h) \in H_\beta$. Let $B_{H_\beta} := \{h \in H_\beta : \|h\|_\beta \leq 1\}$ be the closed unit ball. For $\epsilon \neq 0$ small enough we obtain, by (5.1) and (4.2),

$$\begin{aligned}&\left| \frac{\Psi'(h(0) + \epsilon v(0)) - \Psi'(h(0))}{\epsilon} - \Psi''(h(0))v(0) \right| \\ &= \left| \int_0^1 \Big(\Psi''(h(0) + s\epsilon v(0))v(0) - \Psi''(h(0))v(0) \Big) ds \right| \\ &\leq \int_0^1 |K s \epsilon v(0)^2| ds \leq K \|v\|_{L^\infty(\mathbb{R}_+)}^2 |\epsilon| \leq K C^2 \|v\|_\beta^2 |\epsilon|.\end{aligned}$$

Thus, the latter term converges to zero for $\epsilon \to 0$, uniformly in $v \in B_{H_\beta}$. For $\epsilon \neq 0$ small enough we have, by Hölder's inequality,

$$\begin{aligned}&\int_{\mathbb{R}_+} \left| \frac{d}{dx} \left(\frac{\Psi'(h(x) + \epsilon v(x)) - \Psi'(h(x))}{\epsilon} - \Psi''(h(x))v(x) \right) \right|^2 e^{\beta x} dx \\ &= \int_{\mathbb{R}_+} \left| \frac{d}{dx} \left(\int_0^1 \Big(\Psi''(h(x) + s\epsilon v(x))v(x) - \Psi''(h(x))v(x) \Big) ds \right) \right|^2 e^{\beta x} dx \qquad (5.6) \\ &\leq \int_{\mathbb{R}_+} \int_0^1 \left| \frac{d}{dx} \Big(\Big(\Psi''(h(x) + s\epsilon v(x)) - \Psi''(h(x)) \Big) v(x) \Big) \right|^2 e^{\beta x} ds dx \\ &\leq 2\Delta_1^\epsilon + 2\Delta_2^\epsilon,\end{aligned}$$

where we have set

$$\begin{aligned}\Delta_1^\epsilon &:= \int_{\mathbb{R}_+} \int_0^1 \Big| \Big(\Psi'''(h(x) + s\epsilon v(x))(h'(x) + s\epsilon v'(x)) - \Psi'''(h(x))h'(x) \Big) v(x) \Big|^2 \\ &\qquad \times e^{\beta x} ds dx, \\ \Delta_2^\epsilon &:= \int_{\mathbb{R}_+} \int_0^1 \Big| \Big(\Psi''(h(x) + s\epsilon v(x)) - \Psi''(h(x)) \Big) v'(x) \Big|^2 e^{\beta x} ds dx.\end{aligned}$$

By (5.1) and (4.2) we get

$$\begin{aligned}\Delta_1^\epsilon &\leq 2 \int_{\mathbb{R}_+} \int_0^1 \Big| \Big(\Psi'''(h(x) + s\epsilon v(x)) - \Psi'''(h(x)) \Big) h'(x)v(x) \Big|^2 e^{\beta x} ds dx \\ &\quad + 2 \int_{\mathbb{R}_+} \int_0^1 \big| \Psi'''(h(x) + s\epsilon v(x)) s\epsilon v'(x)v(x) \big|^2 e^{\beta x} ds dx \\ &\leq 2 \int_{\mathbb{R}_+} \int_0^1 |K s\epsilon v(x)^2 h'(x)|^2 e^{\beta x} ds dx + 2 \int_{\mathbb{R}_+} \int_0^1 |K s\epsilon v'(x)v(x)|^2 e^{\beta x} ds dx \\ &\leq 2K^2 \epsilon^2 C^4 \|v\|_\beta^4 \|h\|_\beta^2 + 2K^2 \epsilon^2 C^2 \|v\|_\beta^4\end{aligned}$$

as well as

$$\Delta_2^\epsilon \leq \int_{\mathbb{R}_+} \int_0^1 |K s\epsilon v(x) v'(x)|^2 e^{\beta x} ds dx \leq K^2 \epsilon^2 C^2 \|v\|_\beta^4.$$

Hence, the integral in (5.6) converges to zero for $\epsilon \to 0$ uniformly in $v \in B_{H_\beta}$. □

We can express the HJM drift term defined in (1.3) as

$$\alpha_{\mathrm{HJM}} = m(-\sigma, \Psi'(-\mathcal{I}\sigma)).$$

Combining Lemmas 4.2, 4.4, 5.3 yields $\alpha_{\mathrm{HJM}} \in C^1(H_\beta)$, whence all required conditions from Assumption 2.1 are fulfilled.

6. Affine realizations for Lévy term structure models

We are now ready to present a general result regarding affine realizations for Lévy term structure models, which will be the basis for our subsequent investigations.

In this section, we assume that the volatility σ in the HJMM equation (1.2) is of the form

$$\sigma(h) = \sum_{i=1}^d \Phi_i(h)\lambda_i, \quad h \in H_\beta \tag{6.1}$$

where $d \in \mathbb{N}$ denotes a positive integer, $\Phi_1, \ldots, \Phi_d : H_\beta \to \mathbb{R}$ are functionals and $\lambda_1, \ldots, \lambda_d \in H^0_{\beta'}$ are linearly independent. We assume that $\Phi_i \in C^1(H_\beta; \mathbb{R})$ for $i = 1, \ldots, d$, the inclusion $-\mathcal{I}\sigma(H_\beta) \subset \mathcal{O}$, and that there exist $L, M > 0$ such that for all $i = 1, \ldots, d$ we have

$$|\Phi_i(h_1) - \Phi_i(h_2)| \leq L\|h_1 - h_2\|_\beta \quad \text{for all } h_1, h_2 \in H_\beta,$$
$$|\Phi_i(h)| \leq M \quad \text{for all } h \in H_\beta.$$

Then, Assumption 5.2 is fulfilled.

Note that, in view of Lemma 3.2, this is the most general volatility, which we can have for the HJMM equation (1.2) with an affine realization. The corresponding HJM drift term (5.4) is given by

$$\alpha_{\mathrm{HJM}}(h) = \frac{d}{dx}\Psi\bigg(- \sum_{i=1}^d \Phi_i(h)\Lambda_i \bigg), \quad h \in H_\beta. \tag{6.2}$$

6.1. **Proposition.** *Suppose that for all $i, j = 1, \ldots, d$ we have*

$$D\Phi_i(h)\lambda_j = 0, \quad h \in H_\beta. \tag{6.3}$$

Then, the HJMM equation (1.2) has an affine realization generated by $\langle \lambda_1, \ldots, \lambda_d \rangle$ if and only if we have

$$\lambda_1, \ldots, \lambda_d \in \mathcal{D}\bigg(\frac{d}{dx}\bigg), \tag{6.4}$$

$$\frac{d}{dx}\lambda_i \in \langle \lambda_1, \ldots, \lambda_d \rangle, \quad i = 1, \ldots, d. \tag{6.5}$$

Proof. Suppose the HJMM equation (1.2) has an affine realization generated by $\langle \lambda_1, \ldots, \lambda_d \rangle$ and let $h_0 \in \mathcal{D}(\frac{d}{dx})$ be arbitrary. Applying Corollary 2.13 to the invariant foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$, we have (6.4), and there exist $h_0' \in H_\beta$ and $\mu, \gamma \in C^1(\mathbb{R}^d)$ such that

$$\nu\bigg(h_0 + \sum_{i=1}^d y_i \lambda_i\bigg) = h_0' + \sum_{i=1}^d \mu_i(y)\lambda_i, \quad y \in \mathbb{R}^d \tag{6.6}$$

$$\sigma\bigg(h_0 + \sum_{i=1}^d y_i \lambda_i\bigg) = \sum_{i=1}^d \gamma_i(y)\lambda_i, \quad y \in \mathbb{R}^d. \tag{6.7}$$

By (6.1), (6.3) and (6.7) we have $\gamma \equiv \rho$ for some $\rho \in \mathbb{R}^d$. Taking into account the HJM drift condition (6.2), inserting (6.7) into (6.6) yields

$$\frac{d}{dx}h_0 + \sum_{i=1}^d y_i \frac{d}{dx}\lambda_i + \frac{d}{dx}\Psi\bigg(- \sum_{i=1}^d \rho_i \Lambda_i \bigg) = h_0' + \sum_{i=1}^d \mu_i(y)\lambda_i, \quad y \in \mathbb{R}^d.$$

Differentiating with respect to y_i for $i = 1, \ldots, d$ provides (6.5).

Conversely, suppose conditions (6.4), (6.5) are fulfilled. Then we have (2.21) and there exist $(a_{ij})_{i,j=1,\ldots,d} \subset \mathbb{R}$ such that

$$\frac{d}{dx}\lambda_i = \sum_{j=1}^d a_{ij}\lambda_j, \quad i = 1, \ldots, d. \tag{6.8}$$

Let $h_0 \in \mathcal{D}(\frac{d}{dx})$ be arbitrary. Since we have $\alpha_{\text{HJM}} \in C^1(H_\beta)$ and α_{HJM} is Lipschitz continuous by (5.5), there exists, according to [34, Thm. 6.1.2, Thm. 6.1.5], a unique solution $\psi \in C^1(\mathbb{R}_+; H_\beta)$ with $\psi(\mathbb{R}_+) \subset \mathcal{D}(\frac{d}{dx})$ of the deterministic evolution equation

$$\begin{cases} \psi'(t) &= \frac{d}{dx}\psi(t) + \alpha_{\text{HJM}}(\psi(t)) \\ \psi(0) &= h_0. \end{cases} \tag{6.9}$$

Then, we in particular have (2.20). Define $\mu, \gamma \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ as

$$\mu_i(t,y) := \sum_{j=1}^d a_{ji} y_j, \quad i = 1, \ldots, d \tag{6.10}$$

$$\gamma_i(t,y) := \Phi_i(\psi(t)), \quad i = 1, \ldots, d. \tag{6.11}$$

Due to (6.1) and (6.3), condition (2.23) is satisfied. Using (6.9), (6.3) and (6.8), for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$ we get

$$\begin{aligned} \nu\bigg(\psi(t) + \sum_{i=1}^d y_i\lambda_i\bigg) &= \frac{d}{dx}\bigg(\psi(t) + \sum_{i=1}^d y_i\lambda_i\bigg) + \alpha_{\text{HJM}}\bigg(\psi(t) + \sum_{i=1}^d y_i\lambda_i\bigg) \\ &= \psi'(t) - \alpha_{\text{HJM}}(\psi(t)) + \sum_{i=1}^d y_i \frac{d}{dx}\lambda_i + \alpha_{\text{HJM}}(\psi(t)) \\ &= \psi'(t) + \sum_{i=1}^d \bigg(\sum_{j=1}^d a_{ji}y_j\bigg)\lambda_i = \psi'(t) + \sum_{i=1}^d \mu_i(t,y)\lambda_i, \end{aligned}$$

showing (2.22). By Corollary 2.13, the foliation $(\mathcal{M}_t^{h_0})_{t\geq 0}$ generated by $\langle \lambda_1, \ldots, \lambda_d \rangle$ with parametrization ψ is invariant for the HJMM equation (1.2). □

6.2. **Remark.** *Note that the proof of Proposition 6.1 simultaneously provides the construction of the affine realization. For $h_0 \in \mathcal{D}(\frac{d}{dx})$ the invariant foliation $(\mathcal{M}_t^{h_0})_{t\geq 0}$ generated by $\langle \lambda_1, \ldots, \lambda_d \rangle$ has the parametrization ψ, which is the solution of the deterministic evolution equation (6.9) with $\psi(0) = h_0$. For $h \in \mathcal{M}_{t_0}^{h_0}$ with some $t_0 \in \mathbb{R}_+$ the strong solution $(r_t)_{t\geq 0}$ for (1.2) with $r_0 = h$ is given by (2.27), where the maps $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ for the state process (2.26) are defined in (6.10), (6.11).*

6.3. **Remark.** *Condition (6.3) looks rather ad-hoc, but as we will see in Theorem 8.1 below, we cannot expect more general volatility structures for Lévy term structure models with an affine realization.*

7. Constant volatility

We shall now study the existence of affine realizations for Lévy term structure models in the particular case where the volatility σ in the HJMM equation (1.2) is constant, i.e., we have

$$\sigma \equiv \lambda \tag{7.1}$$

with $\lambda \in H^0_{\beta'}$, $\lambda \neq 0$ and $-\Lambda \in \mathcal{O}$. Then, Assumption 5.2 is fulfilled and the HJM drift term (5.4) is given by

$$\alpha_{\text{HJM}} \equiv \frac{d}{dx}\Psi(-\Lambda). \tag{7.2}$$

7.1. **Proposition.** *The HJMM equation (1.2) has an affine realization if and only if λ is quasi-exponential.*

Proof. Suppose the HJMM equation (1.2) has an affine realization generated by a finite dimensional subspace $V \subset H_\beta$. Set $d := \dim V$. Condition (2.13) from Theorem 2.11 and (7.1) yield that $\lambda \in V$. Set $\lambda_1 := \lambda$ and choose $\lambda_2, \dots, \lambda_d \in V$ such that $\{\lambda_1, \dots, \lambda_d\}$ is a basis of V. Since (6.3) is satisfied due to (7.1), Proposition 6.1 and Lemma 3.6 imply that λ is quasi-exponential.

Conversely, suppose λ is quasi-exponential. By Lemma 3.6 there exists $d \in \mathbb{N}$ such that $\{\lambda, \frac{d}{dx}\lambda, \dots, (\frac{d}{dx})^{d-1}\lambda\}$ is a basis of $\langle (\frac{d}{dx})^i \lambda : i \in \mathbb{N}_0 \rangle$. Then, the functions

$$\lambda_i := \left(\frac{d}{dx}\right)^{i-1}\lambda, \quad i = 1, \dots, d. \tag{7.3}$$

fulfill conditions (6.4), (6.5), and condition (6.3) is satisfied by (7.1). According to Proposition 6.1, the HJMM equation (1.2) has an affine realization generated by $\langle \lambda_1, \dots, \lambda_d \rangle$. □

7.2. **Remark.** *For $h_0 \in \mathcal{D}(\frac{d}{dx})$ the invariant foliation $(\mathcal{M}_t^{h_0})_{t\geq 0}$ is generated by $\langle \lambda_1, \dots, \lambda_d \rangle$ with generating functions defined in (7.3). The parametrization ψ, which is the solution of the deterministic evolution equation (6.9), is given by the variation of constants formula*

$$\begin{aligned} \psi(t) &= S_t h_0 + \int_0^t S_{t-s}\alpha_{\text{HJM}} ds = S_t h_0 + \int_0^t \frac{d}{dx}\Psi(-\Lambda(\bullet + t - s))ds \\ &= S_t h_0 - \Psi(-\Lambda) + S_t\Psi(-\Lambda) = -\Psi(-\Lambda) + S_t(h_0 + \Psi(-\Lambda)), \quad t \geq 0. \end{aligned} \tag{7.4}$$

Combining Proposition 3.3 and relations (7.2), (7.4), the singular set Σ is given by the $(d+1)$-dimensional affine space

$$\Sigma = -\Psi(-\Lambda) + \langle 1, \Lambda_1, \dots, \Lambda_d \rangle,$$

and for each $h_0 \in \mathcal{D}(\frac{d}{dx})$ we have (3.4), (3.5), where

$$t_0 := \inf\{t \geq 0 : S_t(h_0 + \Psi(-\Lambda)) \in \langle 1, \Lambda_1, \dots, \Lambda_d \rangle\} \in [0, \infty]$$

and where $(r_t)_{t\geq 0}$ denotes the strong solution for (1.2) with $r_0 = h_0$.

7.3. **Remark.** *Proposition 7.1 shows that for deterministic volatility structures, we obtain the same condition for the existence of an affine realization as in the classical diffusion case, where the HJMM equation (1.2) is driven by a Brownian motion, namely that λ has to be quasi-exponential, see* [8, Sec. 5] *and* [7, Sec. 4].

8. Constant direction volatility

In this section, we study the existence of affine realizations for Lévy term structure models with constant direction volatility, that is, we assume that the volatility σ in the HJMM equation (1.2) is of the form

$$\sigma(h) = \Phi(h)\lambda, \quad h \in H_\beta \tag{8.1}$$

where $\Phi : H_\beta \to \mathbb{R}$ is a functional and $\lambda \in H_{\beta'}^0$ with $\lambda \neq 0$. We assume that $\Phi \in C^1(H_\beta;\mathbb{R})$, $-\mathcal{I}\sigma(H_\beta) \subset \mathcal{O}$ and that there exist $L, M > 0$ such that

$$|\Phi(h_1) - \Phi(h_2)| \leq L\|h_1 - h_2\|_\beta \quad \text{for all } h_1, h_2 \in H_\beta,$$
$$|\Phi(h)| \leq M \quad \text{for all } h \in H_\beta.$$

Then, Assumption 5.2 is fulfilled. Recall that F denotes the Lévy measure of the driving Lévy process X in (1.2).

8.1. **Theorem.** *Suppose $F(\mathbb{R}) \neq 0$ and $\Phi \not\equiv 0$. Then, the HJMM equation (1.2) has an affine realization if and only if λ is quasi-exponential and we have*

$$D\Phi(h)\left(\frac{d}{dx}\right)^i \lambda = 0, \quad i \in \mathbb{N}_0 \tag{8.2}$$

for all $h \in H_\beta$.

Proof. Assume λ is quasi-exponential and we have (8.2). By Lemma 3.6 there exists $d \in \mathbb{N}$ such that $\{\lambda, \frac{d}{dx}\lambda, \ldots, (\frac{d}{dx})^{d-1}\lambda\}$ is a basis of $\langle (\frac{d}{dx})^i\lambda : i \in \mathbb{N}_0\rangle$. Then, the functions $\lambda_i := (\frac{d}{dx})^{i-1}\lambda$, $i = 1, \ldots, d$ fulfill conditions (6.4), (6.5). Moreover, condition (6.3) is satisfied by (8.2). According to Proposition 6.1, the HJMM equation (1.2) has an affine realization generated by $\langle \lambda_1, \ldots, \lambda_d\rangle$.

Conversely, suppose the HJMM equation (1.2) has an affine realization generated by a finite dimensional subspace $V \subset H_\beta$. Set $d := \dim V$. Since $\Phi \not\equiv 0$ and $\mathcal{D}(\frac{d}{dx})$ is dense in H_β, condition (2.13) from Theorem 2.11 and (8.1) yield that $\lambda \in V$. Set $\lambda_1 := \lambda$ and choose $\lambda_2, \ldots, \lambda_d \in V$ such that $\{\lambda_1, \ldots, \lambda_d\}$ is a basis of V. Now let $h_0 \in \mathcal{D}(\frac{d}{dx})$ be arbitrary. We apply Corollary 2.13 to the invariant foliation $(\mathcal{M}_t^{h_0})_{t\geq 0}$, implying (6.4) and the existence of $h_0' \in H_\beta$ and $\mu \in C^1(\mathbb{R}^m)$, $\gamma \in C^1(\mathbb{R}^m;\mathbb{R})$ such that

$$\nu\bigg(h_0 + \sum_{i=1}^d y_i\lambda_i\bigg) = h_0' + \sum_{i=1}^d \mu_i(y)\lambda_i, \quad y \in \mathbb{R}^d \tag{8.3}$$

$$\sigma\bigg(h_0 + \sum_{i=1}^d y_i\lambda_i\bigg) = \gamma(y)\lambda, \quad y \in \mathbb{R}^d. \tag{8.4}$$

Taking into account the HJM drift condition (5.4), inserting (8.4) into (8.3) gives us

$$\frac{d}{dx}h_0 + \sum_{i=1}^d y_i \frac{d}{dx}\lambda_i + \frac{d}{dx}\Psi(-\gamma(y)\Lambda) = h_0' + \sum_{i=1}^d \mu_i(y)\lambda_i, \quad y \in \mathbb{R}^d.$$

Differentiating with respect to y_k we obtain

$$\frac{d}{dx}\lambda_k - \frac{d}{dx}\Big(\partial_k\gamma(y)\Lambda\Psi'(-\gamma(y)\Lambda)\Big) = \sum_{i=1}^d \partial_k\mu_i(y)\lambda_i, \quad k = 1, \ldots, d$$

for all $y \in \mathbb{R}^d$. Integrating yields

$$\lambda_k - \lambda_k(0) - \sum_{i=1}^d \partial_k\mu_i(y)\Lambda_i - \partial_k\gamma(y)\Lambda\Psi'(-\gamma(y)\Lambda) = 0, \quad k = 1, \ldots, d \tag{8.5}$$

for all $y \in \mathbb{R}^d$. Using Lemma 4.3 there exists an even integer $m \geq 2$ such that the sum

$$\langle \Lambda_1, \ldots, \Lambda_d, \Lambda^2, \ldots, \Lambda^{m-1} \rangle \oplus \langle \Lambda^i : i \geq m \rangle \tag{8.6}$$

is direct. By Taylor's Theorem there exists $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\Psi'(-\gamma(y)\Lambda) = \sum_{n=1}^{m+2} \frac{\Psi^{(n)}(0)}{(n-1)!}(-\gamma(y)\Lambda)^{n-1} + \frac{\Psi^{(m+3)}(\xi(y))}{(m+2)!}(-\gamma(y)\Lambda)^{m+2}, \quad y \in \mathbb{R}.$$

Inserting this identity into (8.5) we get

$$\begin{aligned} &\lambda_k - \lambda_k(0) - \sum_{i=1}^d \partial_k \mu_i(y) \Lambda_i \\ &- \partial_k \gamma(y) \bigg(\sum_{n=1}^{m+2} \frac{\Psi^{(n)}(0)}{(n-1)!}(-\gamma(y))^{n-1}\Lambda^n + \frac{\Psi^{(m+3)}(\xi(y))}{(m+2)!}(-\gamma(y))^{m+2}\Lambda^{m+3} \bigg) = 0 \end{aligned} \tag{8.7}$$

for all $k = 1, \ldots, d$ and $y \in \mathbb{R}^d$. We claim that γ is constant. Suppose, on the contrary, that γ is not constant. Then, there exist $y, z \in \mathbb{R}^d$ with $y \neq z$ and an index $k \in \{1, \ldots, d\}$ such that

$$\begin{cases} \gamma(y) \neq \gamma(z) \text{ and } \gamma(y) \neq -\gamma(z), \\ \gamma(y) \neq 0 \text{ and } \gamma(z) \neq 0, \\ \partial_k \gamma(y) \neq 0 \text{ and } \partial_k \gamma(z) \neq 0. \end{cases} \tag{8.8}$$

From identity (8.7) we obtain the equation

$$\begin{aligned} 0 = &\sum_{i=1}^d \Big(\partial_k \mu_i(z) - \partial_k \mu_i(y) \Big) \Lambda_i \\ &+ \sum_{n=1}^{m+2} \frac{\Psi^{(n)}(0)}{(n-1)!} \Big(\partial_k \gamma(z)(-\gamma(z))^{n-1} - \partial_k \gamma(y)(-\gamma(y))^{n-1} \Big) \Lambda^n \\ &+ \bigg(\frac{\Psi^{(m+3)}(\xi(z))}{(m+2)!} \partial_k \gamma(z)(-\gamma(z))^{m+2} - \frac{\Psi^{(m+3)}(\xi(y))}{(m+2)!} \partial_k \gamma(y)(-\gamma(y))^{m+2} \bigg) \Lambda^{m+3} \end{aligned} \tag{8.9}$$

in $C(\mathbb{R}_+)$. Since we have $F(\mathbb{R}) \neq 0$ by assumption and $m \geq 2$ is even, relations (5.2), (5.3) give us

$$\Psi^{(m)}(0) > 0 \quad \text{and} \quad \Psi^{(m+2)}(0) > 0.$$

Since the sum (8.6) is direct, Lemma 4.3 and the identity (8.9) yield

$$\left(\frac{\gamma(z)}{\gamma(y)} \right)^m = \frac{\partial_k \gamma(y)}{\partial_k \gamma(z)} = \left(\frac{\gamma(z)}{\gamma(y)} \right)^{m+2},$$

which, in view of (8.8), is a contradiction. Consequently, γ is constant. Since $h_0 \in \mathcal{D}(\frac{d}{dx})$ was arbitrary, relations (8.1) and (8.4) show that

$$D\Phi(h)\lambda_i = 0, \quad i = 1, \ldots, d \tag{8.10}$$

for all $h \in \mathcal{D}(\frac{d}{dx})$. Since $\mathcal{D}(\frac{d}{dx})$ is dense in H_β and $\Phi \in C^1(H_\beta; \mathbb{R})$, we deduce (8.10) for all $h \in H_\beta$. Now Proposition 6.1 and Lemma 3.6 yield that λ is quasi-exponential as well as (8.2). □

8.2. **Remark.** *For constant direction volatilities, Theorem 8.1 reveals a discrepancy between Wiener and Lévy driven term structure models concerning the existence of affine realizations.*

- *In the Wiener case, λ has to be quasi-exponential, but Φ is allowed to be an arbitrary functional, see* [8, Sec. 6] *and* [7, Sec. 5]. *In the present Lévy case, we get the additional restriction (8.2) on Φ, which means that at each forward curve $h \in H_\beta$ the functional Φ is constant in direction of the linear space V, which generates the affine realization. However, if Φ is arbitrary, there exists, at least, an affine realization in an approximative sense, see Section 9 below.*
- *In the diffusion case, there are also degenerate examples, like the Cox-Ingersoll-Ross model, which can occur when $D^2\Phi^2(h)(\lambda,\lambda) = 0$, see, e.g.,* [7, Sec. 5.2]. *Then, λ is not quasi-exponential, but satisfies a certain Ricatti equation with an additional quadratic term. Such examples cannot arise for Lévy term structure models, see also the forthcoming Section 10 concerning short rate realizations.*

Often, one considers volatility structures of the form (8.1) with

$$\Phi = \varphi \circ \ell, \tag{8.11}$$

where $\ell \in L(H_\beta;\mathbb{R}^p)$ is a linear operator acting on the forward curve and $\varphi : \mathbb{R}^p \to \mathbb{R}$ for some $p \in \mathbb{N}$. We may think of $\ell_i(h) = \frac{1}{x_i}\int_0^{x_i} h(\eta)d\eta$ (benchmark yields) or $\ell_i(h) = h(x_i)$ (benchmark forward rates).

We assume that $\varphi \in C^1(\mathbb{R}^p;\mathbb{R})$ and that there exist $L, M > 0$ such that

$$|\varphi(y_1) - \varphi(y_2)| \leq L\|y_1 - y_2\|_{\mathbb{R}^p} \quad \text{for all } y_1, y_2 \in \mathbb{R}^p,$$
$$|\varphi(y)| \leq M \quad \text{for all } y \in \mathbb{R}^p.$$

Then, Assumption 5.2 is fulfilled.

8.3. **Corollary.** *Suppose $F(\mathbb{R}) \neq 0$, $\varphi \not\equiv 0$ and that*

$$\left\langle \ell\left(\frac{d}{dx}\right)^i \lambda : i \in \mathbb{N}_0 \right\rangle = \mathbb{R}^p. \tag{8.12}$$

Then, the HJMM equation (1.2) has an affine realization if and only if λ is quasi-exponential and $\Phi \equiv \rho$ for some constant $\rho \neq 0$.

Proof. Assume λ is quasi-exponential and Φ is constant on H_β. Then condition (8.2) is satisfied, and the HJMM equation (1.2) has an affine realization by Theorem 8.1.

Conversely, suppose the HJMM equation (1.2) has an affine realization. By Theorem 8.1, λ is quasi-exponential and we have (8.2). Let $y \in \mathbb{R}^p$ be arbitrary. By (8.12) there exists $h \in H_\beta$ with $\ell(h) = y$. For all $i \in \mathbb{N}_0$ we obtain, by using (8.2),

$$D\varphi(y)\ell\left(\frac{d}{dx}\right)^i\lambda = D(\varphi\circ\ell)(h)\left(\frac{d}{dx}\right)^i\lambda = D\Phi(h)\left(\frac{d}{dx}\right)^i\lambda = 0.$$

In view of (8.12), we deduce that φ is constant on $\mathbb{R}^p$, which completes the proof. □

9. Approximative realizations for constant direction volatility

We have seen in the previous section that for Lévy term structure models with constant direction volatility (8.1) and an arbitrary functional Φ, an affine realization does, in general, not exist.

In this section, we will show that in this situation, with λ being quasi-exponential, there at least exists an affine realization in an approximative sense. The proof of Theorem 8.1 relies on the observation that the derivatives $\Psi^{(n)}(0)$, $n \in \mathbb{N}$ of the cumulant generating function never vanish. Since Ψ is analytic, our idea is to approximate it by its Taylor polynomials.

Let σ be of the form (8.1) such that all assumptions from the beginning of Section 8 are fulfilled. Furthermore, let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial. We define

$$\alpha_p(h) := -\sigma(h) p\Big(- \int_0^\bullet \sigma(h)(\eta) d\eta \Big), \quad h \in H_\beta.$$

Following the argumentation of Section 5, we verify that also the SPDE

$$\begin{cases} dr_t & = & \alpha_p(r_t)dt + \sigma(r_{t-})dX_t \\ r_0 & = & h_0 \end{cases} \tag{9.1}$$

satisfies Assumption 2.1, ensuring existence and uniqueness of weak solutions.

9.1. **Proposition.** *Suppose λ is quasi-exponential. Then, for every polynomial p the SPDE (9.1) has an affine realization.*

Proof. By induction we show that for each $n \in \mathbb{N}_0$ there exists a finite dimensional subspace $V_n \subset \mathcal{D}(\frac{d}{dx})$ with $V_n \subset \bigcup_{\epsilon > 0} H^0_{\beta+\epsilon}$ and

$$\lambda \Lambda^i \in V_n \quad \text{for all } i = 0, \ldots, n \tag{9.2}$$

$$\frac{d}{dx} v \in V_n \quad \text{for all } v \in V_n. \tag{9.3}$$

Since $S_t H^0_{\beta'} \subset H^0_{\beta'}$ for all $t \geq 0$, for $n = 0$ this follows from Lemma 3.6.

For the induction step $n \to n+1$ let $d_n := \dim V_n$. By (9.2) and Lemma 4.3, there exists a basis $\{\lambda_1, \ldots, \lambda_{d_n}\}$ of V_n with $\lambda_i = \lambda \Lambda^{i-1}$ for $i = 1, \ldots, n+1$. We define

$$V_{n+1} := V_n + \langle \lambda_i \Lambda_j : i,j = 1, \ldots, d_n \rangle.$$

By Lemmas 4.2, 4.4 we have $V_{n+1} \subset \mathcal{D}(\frac{d}{dx})$ and $V_{n+1} \subset \bigcup_{\epsilon>0} H^0_{\beta+\epsilon}$ as well as $\lambda \Lambda^i \in V_{n+1}$ for all $i = 0, \ldots, n+1$. Set $d_{n+1} := \dim V_{n+1}$ and choose $\lambda_{d_n+1}, \ldots, \lambda_{d_{n+1}} \in V_{n+1}$ such that $\{\lambda_1, \ldots, \lambda_{d_{n+1}}\}$ is a basis of V_{n+1}. Relation (9.3) implies

$$\frac{d}{dx} \Lambda_i \in \langle 1, \Lambda_1, \ldots, \Lambda_{d_n} \rangle, \quad i = 1, \ldots, d_n. \tag{9.4}$$

By (9.4) and (9.3) we have

$$\frac{d}{dx}(\lambda_i \Lambda_j) = \lambda_i \frac{d}{dx} \Lambda_j + \Lambda_j \frac{d}{dx} \lambda_i \in V_{n+1}, \quad i,j = 1, \ldots, d_n$$

whence we have $\frac{d}{dx} v \in V_{n+1}$ for all $v \in V_{n+1}$.

Now let $n := \deg p$ be the degree of the polynomial

$$p(x) = \sum_{k=0}^n c_k x^k, \quad x \in \mathbb{R}$$

and let $V_n \subset \mathcal{D}(\frac{d}{dx})$ be a finite dimensional subspace such that (9.2) and (9.3) are fulfilled. Set $d := \dim V_n$. By (9.2) and Lemma 4.3 there exists a basis $\{\lambda_1, \ldots, \lambda_d\}$ of V_n with

$$\lambda_i = \lambda \Lambda^{i-1} \quad \text{for } i = 1, \ldots, n+1. \tag{9.5}$$

Moreover, condition (2.21) is satisfied. Let $h_0 \in \mathcal{D}(\frac{d}{dx})$ be arbitrary. We define $\psi \in C^1(\mathbb{R}_+; H_\beta)$ and $\gamma \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ as

$$\psi(t) := S_t h_0,$$

$$\gamma_i(t,y) := \begin{cases} \Phi(\psi(t) + \sum_{j=1}^d y_j \lambda_j), & i = 1 \\ 0, & i = 2, \ldots, d. \end{cases}$$

Then we have (2.20), (2.23) and ψ is the solution of the abstract Cauchy problem

$$\left\{ \begin{array}{rcl} \psi'(t) & = & \frac{d}{dx}\psi(t) \\ \psi(0) & = & h_0. \end{array} \right. \tag{9.6}$$

By (9.3) there exist $(a_{ij})_{i,j=1,\dots,d} \subset \mathbb{R}$ such that

$$\frac{d}{dx}\lambda_i = \sum_{j=1}^{d} a_{ij}\lambda_j, \quad i = 1,\dots,d. \tag{9.7}$$

Furthermore, we define $\mu \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d;\mathbb{R}^d)$ as

$$\mu_i(t,y) := \begin{cases} (-1)^i c_{i-1}\gamma_1(t,y)^i + \sum_{j=1}^{d} a_{ji}y_j, & i = 1,\dots,n+1 \\ \sum_{j=1}^{d} a_{ji}y_j, & i = n+2,\dots,d. \end{cases}$$

Then, by incorporating (9.6), (9.7) and (9.5), for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$ we get

$$\begin{aligned} \nu\bigg(\psi(t) + \sum_{i=1}^{d} y_i\lambda_i\bigg) &= \frac{d}{dx}\psi(t) + \sum_{i=1}^{d} y_i \frac{d}{dx}\lambda_i + \alpha_p\bigg(\psi(t) + \sum_{i=1}^{d} y_i\lambda_i\bigg) \\ &= \psi'(t) + \sum_{i=1}^{d}\bigg(\sum_{j=1}^{d} a_{ji}y_j\bigg)\lambda_i - \gamma_1(t,y)\lambda\sum_{i=0}^{n} c_i(-\gamma_1(t,y))^i\Lambda^i \\ &= \psi'(t) + \sum_{i=1}^{d}\bigg(\sum_{j=1}^{d} a_{ji}y_j\bigg)\lambda_i + \sum_{i=0}^{n}(-1)^{i+1}c_i\gamma_1(t,y)^{i+1}\lambda\Lambda^i \\ &= \psi'(t) + \sum_{i=1}^{d}\mu_i(t,y)\lambda_i, \end{aligned}$$

showing (2.22). By Corollary 2.13, the foliation $(\mathcal{M}_t^{h_0})_{t\geq 0}$ generated by $\langle\lambda_1,\dots,\lambda_d\rangle$ with parametrization ψ is invariant for (9.1). □

9.2. **Theorem.** *Suppose λ is quasi-exponential. Then, for every $n \in \mathbb{N}$ there exists an SPDE*

$$\left\{ \begin{array}{rcl} dr_t^n & = & \alpha_n(r_t^n)dt + \sigma(r_{t-}^n)dX_t \\ r_0^n & = & h_0 \end{array} \right. \tag{9.8}$$

with an affine realization, such that for all $h_0 \in H_\beta$ and $T \in \mathbb{R}_+$ we have

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|r_t - r_t^n\|_\beta^2\bigg] \to 0 \quad \text{for } n \to \infty, \tag{9.9}$$

where $(r_t)_{t\geq 0}$ denotes the weak solution for the HJMM equation (1.2) with $r_0 = h_0$, and $(r_t^n)_{t\geq 0}$ denotes, for every $n \in \mathbb{N}$, the weak solution for (9.8) with $r_0^n = h_0$.

Proof. Since Ψ is analytic, there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of polynomials such that $\Psi_n^{(i)} \to \Psi^{(i)}$ uniformly on $[-N,N]$ for $i = 0,\dots,3$. We define

$$\alpha_n(h) := -\sigma(h)\Psi_n'\bigg(-\int_0^\bullet \sigma(h)(\eta)d\eta\bigg), \quad h \in H_\beta.$$

Since λ is quasi-exponential, for each $n \in \mathbb{N}$ the SPDE (9.8) has an affine realization by Proposition 9.1. Because of the uniform convergence, there exists a constant $K > 0$ such that

$$\sup_{x\in[-N,N]} |\Psi_n^{(i)}(x)| \leq K, \quad n \in \mathbb{N} \text{ and } i = 0,\dots,3. \tag{9.10}$$

Therefore, arguing as in the proof of [21, Prop. 4.5], there exists a joint constant $\tilde{K} > 0$ such that

$$\|\alpha_n(h_1) - \alpha_n(h_2)\|_\beta \leq \tilde{K}\|h_1 - h_2\|_\beta, \quad h_1, h_2 \in H_\beta$$

for each $n \in \mathbb{N}$. Let $h_0 \in H_\beta$ be arbitrary and denote by $(r_t)_{t \geq 0}$ the solution for (1.2) with $r_0 = h_0$. By Assumption 5.2 and Lemma 4.2 there exist constants $M_1, M_2 > 0$ such that

$$\|\sigma(h)\|_\beta \leq M_1, \quad h \in H_\beta$$
$$\|h_1 h_2\|_\beta \leq M_2 \|h_1\|_\beta \|h_2\|_\beta, \quad h_1, h_2 \in H_\beta.$$

Hence, for every $t \in \mathbb{R}_+$ we have

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^t \|\alpha(r_s) - \alpha_n(r_s)\|_\beta^2 ds\bigg] \\ &\leq M_1^2 M_2^2 \mathbb{E}\bigg[\int_0^t \|\Psi'(-\Phi(r_s)\Lambda) - \Psi_n'(-\Phi(r_s)\Lambda)\|_\beta^2 ds\bigg] \to 0 \quad \text{for } n \to \infty. \end{aligned} \tag{9.11}$$

by the uniform convergence $\Psi_n' \to \Psi'$ on $[-N, N]$. By (9.10) and (9.11), [22, Prop. 5.5] applies and yields the desired convergence (9.9). □

10. Short rate realizations

Our previous results have further consequences concerning the existence of short rate realizations for Lévy term structure models, which we shall present in this section.

Let σ be a general volatility structure satisfying Assumption 5.2.

10.1. **Corollary.** *Suppose $F(\mathbb{R}) \neq 0$ and $\sigma \not\equiv 0$. Then, the HJMM equation (1.2) has a one-dimensional affine realization if and only if there exist $\Phi \in C^1(H_\beta; \mathbb{R})$ and $c \in (\frac{\beta'}{2}, \infty) \cup \{0\}$ such that*

$$\sigma(h) = \Phi(h) e^{-c\bullet}, \quad h \in H_\beta \tag{10.1}$$

$$D\Phi(h) e^{-c\bullet} = 0, \quad h \in H_\beta. \tag{10.2}$$

Proof. If we have (10.1) and (10.2), then the HJMM equation (1.2) has, by Proposition 6.1, an affine realization generated by the one-dimensional linear space $\langle e^{-c\bullet} \rangle$.

Conversely, suppose the HJMM equation (1.2) has a one-dimensional affine realization. Then, there exists $\lambda \in H_\beta$ with $\lambda \neq 0$ such that the one-dimensional linear space $\langle \lambda \rangle$ generates the affine realization. By Corollary 2.13 we have $\lambda \in \mathcal{D}(\frac{d}{dx})$, and by Lemma 3.2 there exists $\Phi \in C^1(H_\beta; \mathbb{R})$ such that

$$\sigma(h) = \Phi(h)\lambda, \quad h \in H_\beta.$$

Since $\sigma \not\equiv 0$, we have $\Phi \not\equiv 0$ and, taking into account the assumption $\sigma(H_\beta) \subset H_{\beta'}^0$, the relation $\lambda \in H_{\beta'}^0$. Hence, Theorem 8.1 applies and gives us

$$D\Phi(h)\lambda = 0, \quad h \in H_\beta.$$

Now Proposition 6.1 applies and yields, by taking into account the definition of the norm (4.1), conditions (10.1), (10.2) for some $c \in (\frac{\beta'}{2}, \infty) \cup \{0\}$. □

We shall now consider the situation where the volatility is sensitive with respect to a functional of the forward curve. Suppose that σ is of the form

$$\sigma = \phi \circ \ell, \tag{10.3}$$

where ℓ is a non-trivial linear functional of the forward curve, i.e. $\ell \in H'_\beta$ with $\ell \neq 0$, and $\phi : \mathbb{R} \to H_\beta$. We assume $\phi \in C^1(\mathbb{R}; H_\beta)$, $\phi(\mathbb{R}) \subset H^0_{\beta'}$, $-\mathcal{I}\phi(\mathbb{R}) \subset \mathcal{O}$ and that there exist $L, M > 0$ such that

$$\|\phi(x_1) - \phi(x_2)\|_\beta \leq L|x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R},$$
$$\|\phi(x)\|_\beta \leq M \quad \text{for all } x \in \mathbb{R}.$$

Then, Assumption 5.2 is fulfilled.

10.2. **Corollary.** *Suppose we have $F(\mathbb{R}) \neq 0$ and $\phi \not\equiv 0$. Then, the HJMM equation (1.2) has a one-dimensional affine realization if and only if*

$$\sigma \equiv \rho e^{-c\bullet} \tag{10.4}$$

for some $\rho \neq 0$ and $c \in (\frac{\beta'}{2}, \infty) \cup \{0\}$.

Proof. If σ is of the form (10.4), then the HJMM equation (1.2) has a one-dimensional affine realization by Corollary 10.1.

Conversely, if the HJMM equation (1.2) has a one-dimensional affine realization, then there exist, according to Corollary 10.1, a map $\Phi \in C^1(H_\beta; \mathbb{R})$ and $c \in (\frac{\beta'}{2}, \infty) \cup \{0\}$ such that (10.1) is satisfied. We define the isomorphism $\pi : \mathbb{R} \to \langle e^{-c\bullet} \rangle$, $\pi(y) := y e^{-c\bullet}$ and the map $\varphi := \pi^{-1} \circ \phi : \mathbb{R} \to \mathbb{R}$. By (10.1) and (10.3) we have (8.11). Since $\phi \not\equiv 0$ and $\ell \neq 0$, we have $\varphi \not\equiv 0$ and (8.12). Thus, Corollary 8.3 yields that $\Phi \equiv \rho$ for some $\rho \neq 0$, showing (10.4). □

From the literature, see, e.g., [30, 8, 24], it is well known that for Wiener driven interest rate models the following three types of affine short rate realizations exist:

- The Ho-Lee model.
- The Hull-White extension of the Vasicek model.
- The Hull-White extension of the Cox-Ingersoll-Ross model.

Using Corollary 10.2 with $\ell : H_\beta \to \mathbb{R}$, $\ell(h) := h(0)$ (note that the evaluation of the short rate actually is a continuous linear functional by Theorem 4.1) reveals that, in the Lévy case, only two of these short rate models still exist:

- (10.4) with $c = 0$ is the Ho-Lee model.
- (10.4) with $c \neq 0$ is the Hull-White extension of the Vasicek model.

We close this section with some geometric aspects of the Vasicek model (the Ho-Lee model requires further assumptions on the cumulant generating function Ψ, because otherwise the condition $-\mathcal{I}\sigma(H_\beta) \subset \mathcal{O}$ from Assumption 5.2 will not be fulfilled). For further results on the Lévy driven Vasicek model we refer to [13].

By Remark 7.2, the singular set Σ is given by

$$\Sigma = -\Psi\Big(-\frac{\rho}{c}(1 - e^{-c\bullet})\Big) + \langle 1, e^{-c\bullet} \rangle.$$

For $h_0 = -\Psi(-\frac{\rho}{c}(1 - e^{-c\bullet})) + c_1 1 + c_2 e^{-c\bullet} \in \Sigma$ with $c_1, c_2 \in \mathbb{R}$ the submanifold

$$\mathcal{M} = -\Psi\Big(-\frac{\rho}{c}(1 - e^{-c\bullet})\Big) + c_1 1 + \langle e^{-c\bullet} \rangle \subset \Sigma$$

is an invariant manifold for the HJMM equation (1.2).

For each $h_0 \in \mathcal{D}(\frac{d}{dx})$ the weak solution $(r_t)_{t \geq 0}$ for (1.2) with $r_0 = h_0$ is also a strong solution, and we have (3.4), (3.5), where

$$t_0 := \inf \Big\{ t \geq 0 : S_t\Big(h_0 + \Psi\Big(-\frac{\rho}{c}(1 - e^{-c\bullet})\Big)\Big) \in \langle 1, e^{-c\bullet} \rangle \Big\} \in [0, \infty].$$

Since $\ell(e^{-c\bullet}) = e^{-cx}|_{x=0} = 1$, Proposition 2.8 yields an affine realization with the short rate $\ell(r) = r(0)$ as state process. For each initial curve $h_0 \in \mathcal{D}(\frac{d}{dx})$ the short rate is an Ornstein-Uhlenbeck process, which becomes time-homogeneous for $h_0 \in \Sigma$.

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Vienna Institute of Finance, University of Vienna, and Vienna University of Economics and Business Administration, Heiligenstädter Strasse 46-48, A-1190 Wien, Austria
E-mail address: stefan.tappe@vif.ac.at