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AN ALTERNATIVE APPROACH ON THE EXISTENCE OF AFFINE REALIZATIONS FOR HJM TERM STRUCTURE MODELS

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Abstract. We propose an alternative approach on the existence of affine realizations for HJM interest rate models, which varies from the main references [6, 5, 12] on this topic as follows: On the one hand, we work on a reasonably large space H of forward curves (Björk et al. [6, 5] choose the space H such that the differential operator $\frac{d}{dx}$ is bounded, whence it is rather small), on the other hand, we avoid convenient analysis on Fréchet spaces (this rather technical machinery is used in Filipović and Teichmann [12]) by focusing on affine realizations, which makes our approach rather comprehensible. We also supplement some existence results for particular volatility structures from [6] and provide further insights into the geometry of term structure models.

Key Words: Geometry of interest rate models, invariant foliations, affine realizations, Ricatti equations.

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1. Introduction

The problem concerning the existence and construction of finite dimensional realizations for Heath, Jarrow, Morton (HJM) interest rate models (see [14]) has been studied, for various special cases, in [16, 19, 9, 1, 15, 3, 4, 7, 8], and has finally completely been solved in [6, 5, 12], see also [2] for a survey.

The main idea is to switch to the Musiela parametrization [18], and to consider the forward rates as the solution of a stochastic partial differential equation (SPDE), the so-called HJMM (Heath–Jarrow–Morton–Musiela) equation

$$\begin{cases} dr_t = (\frac{d}{dx} r_t + \alpha_{\mathrm{HJM}}(r_t))dt + \sigma(r_t)dW_t \\ r_0 = h_0 \end{cases} \tag{1.1}$$

on a suitable Hilbert space H of forward curves, where $\frac{d}{dx}$ denotes the differential operator, which is generated by the strongly continuous semigroup $(S_t)_{t\geq 0}$ of shifts.

The implied bond market

$$P(t,T) = \exp\bigg(- \int_t^{T-t} r_t(x)dx \bigg), \quad 0 \leq t \leq T$$

is free of arbitrage if there exists an equivalent (local) martingale measure such that the discounted bond prices

$$\exp\bigg(- \int_0^t r_s(0)ds \bigg) P(t,T), \quad t \in [0,T]$$

are local martingales for all maturities T. If we formulate the HJMM equation (1.1) with respect to such an equivalent martingale measure, then the drift is determined

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by the volatility, i.e. $\alpha_{\mathrm{HJM}} : H \to H$ in (1.1) is given by the HJM drift condition (see [14])

$$\alpha_{\mathrm{HJM}}(h) = \sigma(h) \int_0^\bullet \sigma(h)(\eta) d\eta = \frac{1}{2} \frac{d}{dx} \left(\int_0^\bullet \sigma(h)(\eta) d\eta \right)^2, \quad h \in H. \tag{1.2}$$

Now, we can consider the problem from a geometric point of view, and the existence of a finite dimensional realization just means the existence of an invariant manifold. Applying the Frobenius Theorem we obtain the following necessary and sufficient condition for the existence of an invariant manifold, namely

$$\dim\{\beta(h), \sigma(h)\}_{\mathrm{LA}} < \infty,$$

i.e. the so-called Lie algebra generated by the vector fields

$$h \mapsto \beta(h) := \frac{d}{dx} h + \alpha_{\mathrm{HJM}}(h) - \frac{1}{2} D\sigma(h)\sigma(h)$$

and $h \mapsto \sigma(h)$ must be locally of finite dimension. These are the essential ideas of the mentioned articles [6, 5, 12].

The technical problem with this approach is that the differential operator $\frac{d}{dx}$ is, in general, an unbounded and therefore non-smooth operator. Björk et al. [6, 5] choose the state space H so small such that $\frac{d}{dx}$ becomes bounded. As a consequence, not all forward curves of basic HJM models belong to this space, as for example the forward curves implied by a Cox-Ingersoll-Ross model, see [12, Sec. 1].

Filipović and Teichmann [12] solved this problem by using convenient analysis on Fréchet spaces, developed in [17], which, however, is far from being trivial to carry out. In their paper, they in particular show that any HJM model with a finite dimensional realization necessarily has an affine term structure.

The contribution of the present paper is to propose an alternative approach, which is characterized by the following two major features:

- We work on the Hilbert space H from [11, Sec. 5], which is large enough to capture any reasonable forward curve (as mentioned before, the space from Björk et al. [6, 5] is, in general, too small).
- Simultaneously, this article does not require knowledge about convenient analysis on Fréchet spaces. We avoid this framework by directly focusing on affine realizations, which, due to the mentioned result from [12], does not mean a restriction. This makes our approach rather comprehensible.

Our approach also allows to supplement some existence results for particular volatility structures from [6] (see our comments in Remarks 6.5, 6.9) and to provide further insights into the geometry of term structure models (see our comments in Remarks 6.3, 6.4, 6.11).

The rest of this paper is organized as follows. In Section 2 we provide results on invariant foliations and in Section 3 on affine realizations for general stochastic partial differential equations. In Section 4 we introduce the space of forward curves. Working on this space, we present a result concerning invariant foliations for the HJMM equation (1.1) in Section 5. Using this result, we study the existence of affine realizations for the HJMM equation (1.1) in Section 6.

2. Invariant foliations for general stochastic partial differential equations

In this section, we provide results on invariant foliations for general stochastic partial differential equations, which we will apply to the HJMM equation (1.1) later on.

From now on, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let W be a real-valued Wiener process.

Here, we shall deal with stochastic partial differential equations of the type

$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t \\ r_0 &= h_0 \end{cases} \tag{2.1}$$

on a separable Hilbert space $(H, \| \cdot \|, \langle \cdot, \cdot \rangle)$. In (2.1), the operator $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $(S_t)_{t \geq 0}$ on H with adjoint operator $A^* : \mathcal{D}(A^*) \subset H \to H$. Recall that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense in H, see, e.g., [20, Satz VII.4.6, p. 351].

Concerning the vector fields $\alpha, \sigma : H \to H$ we impose the following condition.

2.1. **Assumption.** *We assume that $\alpha, \sigma \in C^1(H)$ and that there is a constant $L > 0$ such that*

$$\|\alpha(h_1) - \alpha(h_2)\| \leq L\|h_1 - h_2\|, \tag{2.2}$$

$$\|\sigma(h_1) - \sigma(h_2)\| \leq L\|h_1 - h_2\| \tag{2.3}$$

for all $h_1, h_2 \in H$.

The Lipschitz assumptions (2.2), (2.3) ensure that for each $h_0 \in H$ there exists a unique weak solution for (2.1) with $r_0 = h_0$.

2.2. **Definition.** *A subset $U \subset H$ is called* invariant *for (2.1) if for every $h \in U$ we have*

$$\mathbb{P}(r_t \in U) = 1 \quad \textit{for all } t \geq 0$$

where $(r_t)_{t \geq 0}$ denotes the weak solution for (2.1) with $r_0 = h$.

In what follows, let $V \subset H$ be a finite dimensional linear subspace and $d := \dim V$.

2.3. **Definition.** *A family $(\mathcal{M}_t)_{t \geq 0}$ of affine subspaces $\mathcal{M}_t \subset H$, $t \geq 0$ is called a* foliation generated by V *if there exists $\psi \in C^1(\mathbb{R}_+; H)$ such that*

$$\mathcal{M}_t = \psi(t) + V, \quad t \geq 0. \tag{2.4}$$

The map ψ is a parametrization *of the foliation $(\mathcal{M}_t)_{t \geq 0}$.*

2.4. **Remark.** *Note that the parametrization of a foliation $(\mathcal{M}_t)_{t \geq 0}$ generated by V is not unique. However, due to condition (2.4), for two parametrizations ψ_1, ψ_2 we have*

$$\psi_1(t) - \psi_2(t) \in V \quad \textit{for all } t \geq 0.$$

In what follows, let $(\mathcal{M}_t)_{t \geq 0}$ be a foliation generated by V. For every $t \geq 0$ the set $\pi_{V^\perp} \mathcal{M}_t$ consists of exactly one point. Therefore, the map

$$\psi : \mathbb{R}_+ \to H, \quad \psi(t) := \pi_{V^\perp} \mathcal{M}_t$$

is well-defined, and it is the unique parametrization of the foliation $(\mathcal{M}_t)_{t \geq 0}$ such that $\psi(t) \in V^\perp$ for all $t \geq 0$.

2.5. **Definition.** *For each $t \geq 0$ we define the* tangent space

$$T\mathcal{M}_t := \psi'(t) + V.$$

By Remark 2.4, the definition of the tangent is independent of the choice of the parametrization.

2.6. **Definition.** *The foliation $(\mathcal{M}_t)_{t \geq 0}$ of submanifolds is* invariant *for (2.1) if for every $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ we have*

$$\mathbb{P}(r_t \in \mathcal{M}_{t_0 + t}) = 1 \quad \textit{for all } t \geq 0 \tag{2.5}$$

where $(r_t)_{t \geq 0}$ denotes the weak solution for (2.1) with $r_0 = h$.

As we shall see now, an invariant foliation generated by V, provided it exists, is unique.

2.7. **Lemma.** *Let $(\mathcal{M}_t^i)_{t\geq 0}$, $i=1,2$ be two foliations generated by V with $\mathcal{M}_0^1 \cap \mathcal{M}_0^2 \neq \emptyset$, which are invariant for (2.1). Then we have $\mathcal{M}_t^1 = \mathcal{M}_t^2$ for all $t \geq 0$.*

Proof. Choose $h_0 \in \mathcal{M}_0^1 \cap \mathcal{M}_0^2$ and let $(r_t)_{t\geq 0}$ be the weak solution for (2.1) with $r_0 = h_0$. Then we have

$$\pi_{V^\perp}\mathcal{M}_t^1 = \pi_{V^\perp} r_t = \pi_{V^\perp}\mathcal{M}_t^2, \quad t \geq 0$$

which completes the proof. □

2.8. **Proposition.** *Suppose the foliation $(\mathcal{M}_t)_{t\geq 0}$ of submanifolds is invariant for (2.1) and let $\ell \in L(H;\mathbb{R}^d)$ be a continuous linear operator with $\ell(V) = \mathbb{R}^d$. Then, for every $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ we have almost surely*

$$r_t = \pi_{V^\perp}\mathcal{M}_{t_0+t} + \ell^{-1}(\ell(r_t) - \ell\pi_{V^\perp}\mathcal{M}_{t_0+t}), \quad t \geq 0 \tag{2.6}$$

where $(r_t)_{t\geq 0}$ denotes the weak solution for (2.1) with $r_0 = h$, and (2.6) is the decomposition of $(r_t)_{t\geq 0}$ according to $V^\perp \oplus V$.

Proof. By condition (2.5) we obtain almost surely

$$r_t = \pi_{V^\perp} r_t + \pi_V r_t = \pi_{V^\perp}\mathcal{M}_{t_0+t} + \pi_V r_t, \quad t \geq 0. \tag{2.7}$$

Therefore we obtain almost surely

$$\pi_V r_t = r_t - \pi_{V^\perp}\mathcal{M}_{t_0+t} = \ell^{-1}(\ell(r_t) - \ell(\pi_{V^\perp}\mathcal{M}_{t_0+t})), \quad t \geq 0. \tag{2.8}$$

Inserting (2.8) into (2.7), we arrive at (2.6). □

2.9. **Remark.** *If the foliation $(\mathcal{M}_t)_{t\geq 0}$ is invariant for (2.1), then for every continuous linear operator $\ell \in L(H;\mathbb{R}^d)$ with $\ell(V) = \mathbb{R}^d$ the decomposition (2.6) provides a realization of the solution $(r_t)_{t\geq 0}$ by means of the finite dimensional process $\ell(r)$.*

We shall now approach our main result of this section, Theorem 2.11 below, which provides consistency conditions for invariance of the foliation $(\mathcal{M}_t)_{t\geq 0}$.

2.10. **Lemma.** *There exist $\zeta_1, \dots, \zeta_d \in \mathcal{D}(A^*)$ and an isomorphism $\phi : \mathbb{R}^d \to V$ such that*

$$\phi(\langle \zeta, h \rangle) = h \quad \textit{for all } h \in V, \tag{2.9}$$

where we use the notation $\langle \zeta, h \rangle := (\langle \zeta_1, h \rangle, \dots, \langle \zeta_d, h \rangle) \in \mathbb{R}^d$.

Proof. By the Gram-Schmidt method, there exists an orthonormal basis $\{e_1, \dots, e_d\}$ of V. Since $\mathcal{D}(A^*)$ is dense in H, there exist $\zeta_1, \dots, \zeta_d \in \mathcal{D}(A^*)$ with $\|\zeta_i - e_i\| < 2^{-d}$ for $i = 1, \dots, d$. Hence, we obtain

$$|\langle \zeta_i, e_j \rangle| \leq |\langle e_i, e_j \rangle| + |\langle \zeta_i - e_i, e_j \rangle| < 2^{-d}$$

for all $i, j = 1, \dots, d$ with $i \neq j$ and

$$|\langle \zeta_i, e_i \rangle| \geq |\langle e_i, e_i \rangle| - |\langle \zeta_i - e_i, e_i \rangle| > 1 - 2^{-d}$$

for all $i = 1, \dots, d$. Thus, we have

$$\sum_{\substack{j=1 \\ j\neq i}}^d |\langle \zeta_i, e_j \rangle| < (d-1)2^{-d} < (2^d - 1)2^{-d} = 1 - 2^d < |\langle \zeta_i, e_i \rangle|$$

for all $i = 1, \dots, d$, and hence, due to the Theorem of Gerschgorin, the $(d \times d)$-matrix

$$B := \begin{pmatrix} \langle \zeta_1, e_1 \rangle & \cdots & \langle \zeta_1, e_d \rangle \\ \vdots & & \vdots \\ \langle \zeta_d, e_1 \rangle & \cdots & \langle \zeta_d, e_d \rangle \end{pmatrix}$$

is invertible. Let $\psi : V \to \mathbb{R}^d$ be the isomorphism

$$\psi(h) := (\langle e_1, h\rangle, \dots, \langle e_d, h\rangle), \quad h \in V.$$

Then, the isomorphism $B\psi : V \to \mathbb{R}^d$ has the representation

$$B\psi(h) = (\langle \zeta_1, h\rangle, \dots, \langle \zeta_d, h\rangle), \quad h \in V.$$

Defining the isomorphism $\phi := (B\psi)^{-1} : \mathbb{R}^d \to V$ completes the proof. □

Now, let $\psi \in C^1(\mathbb{R}_+; H)$ be a parametrization of $(\mathcal{M}_t)_{t\geq 0}$ and let $\phi : \mathbb{R}^d \to V$ be an isomorphism as in Lemma 2.10. We define $\tilde{\alpha}, \tilde{\sigma} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ as

$$\begin{aligned}
\tilde{\alpha}(t,z) &:= \langle A^*\zeta, \psi(t) + \phi(z)\rangle + \langle \zeta, \alpha(\psi(t) + \phi(z)) - \psi'(t)\rangle, \\
\tilde{\sigma}(t,z) &:= \langle \zeta, \sigma(\psi(t) + \phi(z))\rangle.
\end{aligned}$$

By Assumption 2.1 we have $\tilde{\alpha}, \tilde{\sigma} \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ and there exists a constant $K > 0$ such that

$$\begin{aligned}
\|\tilde{\alpha}(t,z_1) - \tilde{\alpha}(t,z_2)\|_{\mathbb{R}^d} &\leq K\|z_1 - z_2\|_{\mathbb{R}^d} \\
\|\tilde{\sigma}(t,z_1) - \tilde{\sigma}(t,z_2)\|_{\mathbb{R}^d} &\leq K\|z_1 - z_2\|_{\mathbb{R}^d}
\end{aligned}$$

for all $t \in \mathbb{R}_+$ and all $z_1, z_2 \in \mathbb{R}^d$. Thus, for each $t_0 \in \mathbb{R}_+$ and each $z_0 \in \mathbb{R}^d$ there exists a unique strong solution for

$$\left\{ \begin{array}{rcl} dZ_t &=& \tilde{\alpha}(t_0 + t, Z_t)dt + \tilde{\sigma}(t_0 + t, Z_t)dW_t \\ Z_0 &=& z_0. \end{array} \right. \tag{2.10}$$

We define the vector field $\nu : \mathcal{D}(A) \to H$ as

$$\nu(h) := Ah + \alpha(h), \quad h \in \mathcal{D}(A).$$

Here is our main result concerning invariance of the foliation $(\mathcal{M}_t)_{t\geq 0}$ for the SPDE (2.1).

2.11. **Theorem.** *The foliation $(\mathcal{M}_t)_{t\geq 0}$ is an invariant foliation for (2.1) if and only if for all $t \geq 0$ we have*

$$\mathcal{M}_t \subset \mathcal{D}(A), \tag{2.11}$$

$$\nu(h) \in T\mathcal{M}_t, \quad h \in \mathcal{M}_t \tag{2.12}$$

$$\sigma(h) \in V, \quad h \in \mathcal{M}_t. \tag{2.13}$$

If the previous conditions are satisfied, the map

$$\mathbb{R}_+ \to H, \quad t \mapsto A(\pi_{V^\perp}\mathcal{M}_t) \tag{2.14}$$

is continuous, and for every $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ the weak solution for (2.1) with $r_0 = h$ is also a strong solution.

Proof. "⇒": Let $t_0 \in \mathbb{R}_+$ and $h_0 \in V$ be arbitrary. Then we have $h := \psi(t_0) + h_0 \in \mathcal{M}_{t_0}$. Let $(r_t)_{t\geq 0}$ be the weak solution for (2.1) with $r_0 = h$ and set $z_0 := \langle \zeta, h_0\rangle$. Since $\zeta_1, \dots, \zeta_d \in \mathcal{D}(A^*)$ and $(\mathcal{M}_t)_{t\geq 0}$ is an invariant foliation for (2.1), we obtain, by using (2.9),

$$\begin{aligned}\langle \zeta, r_t - \psi(t_0+t)\rangle &= \langle \zeta, h - \psi(t_0)\rangle + \int_0^t (\langle A^*\zeta, r_s\rangle + \langle \zeta, \alpha(r_s) - \psi'(t_0+s)\rangle)ds \\ &\quad + \int_0^t \langle \zeta, \sigma(r_s)\rangle dW_s \\ &= \langle \zeta, h_0\rangle + \int_0^t \tilde{\alpha}(t_0+s, \langle \zeta, r_s - \psi(t_0+s)\rangle)ds \\ &\quad + \int_0^t \tilde{\sigma}(t_0+s, \langle \zeta, r_s - \psi(t_0+s)\rangle)dW_s.\end{aligned}$$

This identity shows that almost surely

$$Z_t = \langle \zeta, r_t - \psi(t_0+t)\rangle, \quad t \geq 0$$

where $(Z_t)_{t\geq 0}$ denotes the strong solution for (2.10) with $Z_0 = z_0$. By (2.9), we have almost surely

$$\phi(Z_t) = r_t - \psi(t_0+t), \quad t \geq 0.$$

Let $\xi \in \mathcal{D}(A^*)$ be arbitrary. We obtain, by Itô's formula and applying the linear functional $\langle \xi, \cdot\rangle$ afterwards,

(2.15)
$$\begin{aligned}\langle \xi, r_t - \psi(t_0+t)\rangle = \langle \xi, \phi(Z_t)\rangle &= \langle \xi, h_0 - \psi(t_0)\rangle + \int_0^t \langle \xi, \phi(\tilde{\alpha}(t_0+s, Z_s))\rangle ds \\ &\quad + \int_0^t \langle \xi, \phi(\tilde{\sigma}(t_0+s, Z_s))\rangle dW_s.\end{aligned}$$

Since $(r_t)_{t\geq 0}$ is a weak solution for (2.1) with $r_0 = h_0$, we have

(2.16)
$$\begin{aligned}\langle \xi, r_t - \psi(t_0+t)\rangle &= \langle \xi, h_0 - \psi(t_0)\rangle + \int_0^t (\langle A^*\xi, r_s\rangle + \langle \xi, \alpha(r_s) - \psi'(t_0+s)\rangle)ds \\ &\quad + \int_0^t \langle \xi, \sigma(r_s)\rangle dW_s.\end{aligned}$$

Combining (2.15) and (2.16) we get

$$\begin{aligned}0 &= \int_0^t (\langle A^*\xi, r_s\rangle + \langle \xi, \alpha(r_s) - \psi'(t_0+s) - \phi(\tilde{\alpha}(t_0+s, Z_s))\rangle)ds \\ &\quad + \int_0^t \langle \xi, \sigma(r_{s-}) - \phi(\tilde{\sigma}(t_0+s, Z_s))\rangle dW_s.\end{aligned} \tag{2.17}$$

Therefore, all integrands in (2.17) vanish and, since $\xi \in \mathcal{D}(A^*)$ was arbitrary, setting $s = 0$ yields $\psi(t_0) + h_0 \in \mathcal{D}(A)$, proving (2.11) and the identities

$$\nu(\psi(t_0) + h_0) = \psi'(t_0) + \phi(\tilde{\alpha}(t_0, z_0)) \in T\mathcal{M}_{t_0}, \tag{2.18}$$

$$\sigma(\psi(t_0) + h_0) = \phi(\tilde{\sigma}(t_0, z_0)) \in V \tag{2.19}$$

which show (2.12), (2.13). Furthermore, identity (2.18) proves the continuity of the map defined in (2.14).

"$\Leftarrow$": Let $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ be arbitrary. There exists a unique $z_0 \in \mathbb{R}^d$ such that $h = \psi(t_0) + \phi(z_0)$. Let $(Z_t)_{t\geq 0}$ be the strong solution for (2.10) with $Z_0 = z_0$.

Itô's formula yields, by using (2.11)–(2.13) and (2.9),

$$\begin{aligned}\psi(t_0+t)+\phi(Z_t) &= \psi(t_0)+\phi(z_0)+\int_0^t(\psi'(t_0+s)+\phi(\tilde{\alpha}(t_0+s,Z_s)))ds\\ &+\int_0^t\phi(\tilde{\sigma}(t_0+s,Z_s))dW_s\\ &= h+\int_0^t(\psi'(t_0+s)+\phi(\langle\zeta,\nu(\psi(t_0+s)+\phi(Z_s))-\psi'(t_0+s)\rangle))ds\\ &+\int_0^t\phi(\langle\zeta,\sigma(\psi(t_0+s)+\phi(Z_s))\rangle)dW_s\\ &= h+\int_0^t(A(\psi(t_0+s)+\phi(Z_s))+\alpha(\psi(t_0+s)+\phi(Z_s)))ds\\ &+\int_0^t\sigma(\psi(t_0+s)+\phi(Z_s))dW_s,\quad t\geq 0.\end{aligned}$$

By the uniqueness of solutions for (2.1) we obtain almost surely

$$r_t=\psi(t_0+t)+\phi(Z_t)\in\mathcal{M}_{t_0+t},\quad t\geq 0$$

where $(r_t)_{t\geq 0}$ denotes the weak solution for (2.1) with $r_0=h$, whence $(\mathcal{M}_t)_{t\geq 0}$ is an invariant foliation, and we get that $(r_t)_{t\geq 0}$ is also a strong solution. □

2.12. **Remark.** *Note that (2.11)–(2.13) are consistency conditions on the tangent spaces (for related results see, e.g., [10]). Since the foliation $(\mathcal{M}_t)_{t\geq 0}$ consists of affine manifolds, we do not need a Stratonovich correction term for the drift.*

Now, we express the consistency conditions from Theorem 2.11 by means of a coordinate system. Let $\psi\in C^1(\mathbb{R}_+;H)$ be a parametrization of $(\mathcal{M}_t)_{t\geq 0}$ and let $\{\lambda_1,\dots,\lambda_d\}$ be a basis of V.

2.13. **Corollary.** *The following statements are equivalent:*

1. $(\mathcal{M}_t)_{t\geq 0}$ *is an invariant foliation for (2.1).*
2. *We have*

$$\psi(\mathbb{R}_+)\subset\mathcal{D}(A), \tag{2.20}$$

$$\lambda_1,\dots,\lambda_d\in\mathcal{D}(A) \tag{2.21}$$

and there exist $\mu,\gamma:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ *such that*

$$\nu\Big(\psi(t)+\sum_{i=1}^d y_i\lambda_i\Big)=\psi'(t)+\sum_{i=1}^d\mu_i(t,y)\lambda_i,\quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^d \tag{2.22}$$

$$\sigma\Big(\psi(t)+\sum_{i=1}^d y_i\lambda_i\Big)=\sum_{i=1}^d\gamma_i(t,y)\lambda_i,\quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^d. \tag{2.23}$$

If the previous conditions are satisfied, μ and γ are uniquely determined, we have $\mu,\gamma\in C^{0,1}(\mathbb{R}_+\times\mathbb{R}^d;\mathbb{R}^d)$, there exists a constant $K>0$ such that

$$\|\mu(t,y_1)-\mu(t,y_2)\|_{\mathbb{R}^d}\leq K\|y_1-y_2\|_{\mathbb{R}^d} \tag{2.24}$$

$$\|\gamma(t,y_1)-\gamma(t,y_2)\|_{\mathbb{R}^d}\leq K\|y_1-y_2\|_{\mathbb{R}^d} \tag{2.25}$$

for all $t\in\mathbb{R}_+$ and $y_1,y_2\in\mathbb{R}^d$, and for every $t_0\in\mathbb{R}_+$ and $h\in\mathcal{M}_{t_0}$ the weak solution for (2.1) with $r_0=h$ is also a strong solution.

Proof. The asserted equivalence follows from Theorem 2.11. By the linear independence of $\lambda_1, \dots, \lambda_d$, the mappings μ and γ are uniquely determined. Denoting by $\pi : \mathbb{R}^d \to V$ the isomorphism $\pi(y) := \sum_{i=1}^d y_i \lambda_i$, we can express them as

$$\mu(t,y) = \pi^{-1}\bigg(\nu\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i \bigg) - \psi'(t) \bigg),$$

$$\gamma(t,y) = \pi^{-1}\bigg(\sigma\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i \bigg) \bigg).$$

Since the map defined in (2.14) is continuous by Theorem 2.11, we have $\mu, \gamma \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ and (2.24), (2.25) by virtue of Assumption 2.1. □

Suppose the foliation $(\mathcal{M}_t)_{t \geq 0}$ is invariant for (2.1). We shall now identify the underlying coordinate process Y. Let $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$ be arbitrary. There exists a unique $y \in \mathbb{R}^d$ such that $h = \psi(t_0) + \sum_{i=1}^d y_i \lambda_i$. Taking into account (2.24), (2.25), we let $(Y_t)_{t \geq 0}$ be the strong solution for

$$\left\{ \begin{array}{rcl} dY_t & = & \mu(t_0 + t, Y_t)dt + \gamma(t_0 + t, Y_t)dW_t \\ Y_0 & = & y, \end{array} \right. \tag{2.26}$$

where $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are given by (2.22), (2.23). By Itô's formula, the process

$$r_t = \psi(t_0 + t) + \sum_{i=1}^d Y_t^i \lambda_i, \quad t \geq 0 \tag{2.27}$$

is the strong solution for (2.1) with $r_0 = h$.

2.14. **Remark.** *If we think of interest rate models, the state process Y has no direct economic interpretation. Proposition 2.8 shows that for any continuous linear operator $\ell : H \to \mathbb{R}^d$ with $\ell(V) = \mathbb{R}^d$ we can choose $\ell(r)$ as state process. We may think of $\ell_i(h) = \frac{1}{x_i} \int_0^{x_i} h(\eta) d\eta$ (benchmark yields) or $\ell_i(h) = h(x_i)$ (benchmark forward rates). We refer to* [5, Sec. 7], [4, Prop. 5.1], [6, Thm. 3.3], [8, Prop. 2], [9, Sec. 5] *for related results.*

3. Affine realizations for general stochastic partial differential equations

The results of the previous section lead to the following definition of an affine realization.

3.1. **Definition.** *Let $V \subset H$ be a finite dimensional linear subspace. The SPDE (2.1) has an* affine realization generated by V *if for each $h_0 \in \mathcal{D}(A)$ there exists a foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ generated by V with $h_0 \in \mathcal{M}_0^{h_0}$, which is invariant for (2.1).*

We call $d := \dim V$ the *dimension* of the affine realization.

3.2. **Lemma.** *Let $d \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_d \in H$ be linearly independent. Suppose the SPDE (2.1) has a d-dimensional affine realization generated by $V = \langle \lambda_1, \dots, \lambda_d \rangle$. Then, there exist $\Phi_1, \dots, \Phi_d \in C^1(H; \mathbb{R})$ such that*

$$\sigma(h) = \sum_{i=1}^d \Phi_i(h) \lambda_i, \quad h \in H. \tag{3.1}$$

Proof. Relation (2.13) from Theorem 2.11 yields $\sigma(h) \in V$ for all $h \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in H and V is closed, we obtain $\sigma(h) \in V$ for all $h \in H$. Hence, there exist $\Phi_1, \dots, \Phi_d : H \to \mathbb{R}$ such that (3.1) is satisfied. Since $\sigma \in C^1(H)$, we have $\Phi_1, \dots, \Phi_d \in C^1(H; \mathbb{R})$. □

Suppose the SPDE (2.1) has an affine realization generated by a finite dimensional subspace $V \subset H$. Then, for each $h_0 \in \mathcal{D}(A)$ the foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ is uniquely determined by Lemma 2.7. We define the *singular set* Σ as

$$\begin{aligned} \Sigma &= \{h_0 \in \mathcal{D}(A) : \mathcal{M}_0^{h_0} = \mathcal{M}_t^{h_0} \text{ for all } t \geq 0\} \\ &= \{h_0 \in \mathcal{D}(A) : h_0 + V \text{ is an invariant manifold}\}. \end{aligned}$$

A consequence of this definition is the identity

$$\Sigma + V = \Sigma. \tag{3.2}$$

In particular, Σ is an invariant set for (2.1).

3.3. **Proposition.** *Suppose the SPDE (2.1) has an affine realization generated by V. Then, the singular set Σ is given by*

$$\Sigma = \{h_0 \in \mathcal{D}(A) : \nu(h_0) \in V\}, \tag{3.3}$$

for each $h_0 \in \mathcal{D}(A)$ the weak solution $(r_t)_{t \geq 0}$ for (2.1) with $r_0 = h_0$ is also a strong solution, and we have

$$\mathbb{P}(r_t \notin \Sigma) = 1, \quad t \in [0, t_0) \tag{3.4}$$

$$\mathbb{P}(r_t \in \Sigma) = 1, \quad t \in [t_0, \infty) \tag{3.5}$$

where we have set

$$t_0 := \inf\{t \geq 0 : \mathcal{M}_t^{h_0} \subset \Sigma\} \in [0, \infty].$$

Proof. Let $h_0 \in \mathcal{D}(A)$ be arbitrary. By condition (2.12) of Theorem 2.11 we have $\nu(h_0) \in V$ if and only if $\nu(h) \in V$ for all $h \in h_0 + V$, which means that $h_0 + V$ is an invariant manifold, proving (3.3). By Theorem 2.11, the weak solution $(r_t)_{t \geq 0}$ for (2.1) with $r_0 = h_0$ is also a strong solution, and, by taking into account (3.2), we obtain (3.4) and (3.5). □

3.4. **Remark.** *Note that the time t_0 is deterministic. By (3.2), for any parametrization ψ of the foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ it is given by*

$$t_0 = \inf\{t \geq 0 : \psi(t) \in \Sigma\}.$$

We observe the following dichotomic behaviour of the solutions for (2.1). Up to time t_0, the solution proceeds outside the singular Σ, afterwards it stays in Σ, and therefore even on an affine manifold. In particular, if $t_0 = 0$ we have $\mathbb{P}(r_t \in \Sigma) = 1$ for all $t \geq 0$, and if $t_0 = \infty$ we have $\mathbb{P}(r_t \notin \Sigma) = 1$ for all $t \geq 0$.

For our later investigations on the existence of affine realizations, quasi-exponential functions (cf. [6, Sec. 5]), which we shall now introduce in this general context, will play an important role. Inductively, we define the domains

$$\mathcal{D}(A^n) := \{h \in \mathcal{D}(A^{n-1}) : A^{n-1}h \in \mathcal{D}(A)\}, \quad n \geq 2$$

as well as the intersection

$$\mathcal{D}(A^\infty) := \bigcap_{n \in \mathbb{N}} \mathcal{D}(A^n).$$

3.5. **Definition.** *An element $h \in \mathcal{D}(A^\infty)$ is called* quasi-exponential *if*

$$\dim\langle A^n h : n \in \mathbb{N}_0 \rangle < \infty. \tag{3.6}$$

3.6. **Lemma.** *Let $h \in H$, $h \neq 0$ be arbitrary. The following statements are equivalent.*

1. *h is quasi-exponential.*
2. *We have $h \in \mathcal{D}(A^\infty)$ and there exists $d \in \mathbb{N}$ such that $\{h, Ah, \ldots, A^{d-1}h\}$ is a basis of $\langle A^n h : n \in \mathbb{N}_0 \rangle$.*

(3) *There exists a finite dimensional subspace $V \subset \mathcal{D}(A)$ with $h \in V$ such that*

$$Av \in V \quad \text{for all } v \in V. \tag{3.7}$$

Proof. (1) $\Rightarrow$ (2): Since $h \neq 0$ is quasi-exponential, there exists a minimal integer $d \in \mathbb{N}$ such that $h, Ah, \ldots, A^{d-1}h$ are linearly independent. By induction, we show that

$$A^n h \in \langle h, Ah, \ldots, A^{d-1}h \rangle \quad \text{for all } n \geq d,$$

whence $\{h, Ah, \ldots, A^{d-1}h\}$ is a basis of $\langle A^n h : n \in \mathbb{N}_0 \rangle$.

(2) $\Rightarrow$ (3): The finite dimensional subspace $V = \langle h, Ah, \ldots, A^{d-1}h \rangle$ has the desired properties.

(3) $\Rightarrow$ (1): Using (3.7), by induction, for each $n \in \mathbb{N}$ we have $h \in \mathcal{D}(A^n)$ and $A^n h \in V$, which yields $h \in \mathcal{D}(A^\infty)$ and (3.6), whence h is quasi-exponential. □

4. The space of forward curves

In this section, we define the space of forward curves, on which we will study the HJMM equation (1.1) in the forthcoming sections. These spaces have been introduced in [11, Sec. 5].

We fix an arbitrary constant $\beta > 0$. Let H_β be the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \right)^{\frac{1}{2}} < \infty. \tag{4.1}$$

Let $(S_t)_{t \geq 0}$ be the shift semigroup on H_β defined by $S_t h := h(t + \cdot)$ for $t \in \mathbb{R}_+$.

Since forward curves should flatten for large time to maturity x, the choice of H_β is reasonable from an economic point of view.

4.1. **Theorem.** *Let $\beta > 0$ be arbitrary.*

(1) *The space $(H_\beta, \| \cdot \|_\beta)$ is a separable Hilbert space.*
(2) *For each $x \in \mathbb{R}_+$, the point evaluation $h \mapsto h(x) : H_\beta \to \mathbb{R}$ is a continuous linear functional.*
(3) *$(S_t)_{t \geq 0}$ is a C_0-semigroup on H_β with infinitesimal generator $\frac{d}{dx} : \mathcal{D}(\frac{d}{dx}) \subset H_\beta \to H_\beta$, $\frac{d}{dx} h = h'$, and domain*

$$\mathcal{D}\left(\frac{d}{dx}\right) = \{h \in H_\beta : h' \in H_\beta\}.$$

(4) *Each $h \in H_\beta$ is continuous, bounded and the limit $h(\infty) := \lim_{x \to \infty} h(x)$ exists.*
(5) *$H_\beta^0 := \{h \in H_\beta : h(\infty) = 0\}$ is a closed subspace of H_β.*
(6) *There exists a universal constant $C > 0$, only depending on β, such that for all $h \in H_\beta$ we have the estimate*

$$\|h\|_{L^\infty(\mathbb{R}_+)} \leq C \|h\|_\beta. \tag{4.2}$$

(7) *For each $\beta' > \beta$, we have $H_{\beta'} \subset H_\beta$ and the relation*

$$\|h\|_\beta \leq \|h\|_{\beta'}, \quad h \in H_{\beta'}. \tag{4.3}$$

Proof. Note that H_β is the space H_w from [11, Sec. 5.1] with weight function $w(x) = e^{\beta x}$, $x \in \mathbb{R}_+$. Hence, the first six statements follow from [11, Thm. 5.1.1, Cor. 5.1.1]. For each $\beta' > \beta$, the observation

$$\int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \leq \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta' x} dx, \quad h \in H_{\beta'}$$

shows $H_{\beta'} \subset H_\beta$ and (4.3). □

4.2. **Lemma.** *The following statements are valid.*

(1) *For all $h, g \in H_\beta$ we have $hg \in H_\beta$ and the multiplication map $m : H_\beta \times H_\beta \to H_\beta$ defined as $m(h,g) := hg$ is a continuous, bilinear operator.*
(2) *For all $h, g \in \mathcal{D}(\frac{d}{dx})$ we have $hg \in \mathcal{D}(\frac{d}{dx})$.*

Proof. The function hg is absolutely continuous, because h and g are absolutely continuous and bounded, see Theorem 4.1. By estimate (4.2) we obtain

$$\begin{aligned}\|hg\|_\beta^2 &= |h(0)|^2|g(0)|^2 + \int_{\mathbb{R}_+} |h(x)g'(x) + g(x)h'(x)|^2 e^{\beta x} dx \\ &\leq \|h\|^2_{L^\infty(\mathbb{R}_+)}\|g\|^2_{L^\infty(\mathbb{R}_+)} + 2\|h\|^2_{L^\infty(\mathbb{R}_+)} \int_{\mathbb{R}_+} |g'(x)|^2 e^{\beta x} dx \\ &\quad + 2\|g\|^2_{L^\infty(\mathbb{R}_+)} \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \\ &\leq C^4\|h\|_\beta^2\|g\|_\beta^2 + 2C^2\|h\|_\beta^2\|g\|_\beta^2 + 2C^2\|g\|_\beta^2\|h\|_\beta^2 < \infty.\end{aligned}$$

Hence, we have $hg \in H_\beta$ and the estimate

$$\|m(h,g)\|_\beta \leq \sqrt{C^4 + 4C^2}\|h\|_\beta\|g\|_\beta,$$

proving that m is a continuous, bilinear operator.

If $h, g \in \mathcal{D}(\frac{d}{dx})$, we have $hg \in C^1(\mathbb{R}_+)$ with $(hg)' = h'g + hg'$, whence $hg \in \mathcal{D}(\frac{d}{dx})$ by the first statement. □

For $\lambda \in H_\beta$ we define $\Lambda := \mathcal{I}\lambda := \int_0^\bullet \lambda(\eta)d\eta$, which belongs to $C(\mathbb{R}_+)$, the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$.

4.3. **Lemma.** *Let $0 < \beta < \beta'$ be arbitrary real numbers. For each $\lambda \in H^0_{\beta'}$ we have $\Lambda \in H_\beta$ and the map $\mathcal{I} : H^0_{\beta'} \to H_\beta$ is a continuous linear operator.*

Proof. Let $\lambda \in H^0_{\beta'}$ be arbitrary. Then $\mathcal{I}\lambda$ is absolutely continuous. Since $\mathcal{I}\lambda(0) = 0$, using Hölder's inequality, we obtain

$$\begin{aligned}\|\mathcal{I}\lambda\|_\beta^2 &= \int_{\mathbb{R}_+} \lambda(x)^2 e^{\beta x} dx = \int_{\mathbb{R}_+} \left(\int_x^\infty \lambda'(y) e^{\frac{1}{2}\beta' y} e^{-\frac{1}{2}\beta' y} dy \right)^2 e^{\beta x} dx \\ &\leq \int_{\mathbb{R}_+} \left(\int_x^\infty \lambda'(y)^2 e^{\beta' y} dy \right) \left(\int_x^\infty e^{-\beta' y} dy \right) e^{\beta x} dx \\ &\leq \|\lambda\|_{\beta'}^2 \int_{\mathbb{R}_+} \frac{1}{\beta'} e^{-(\beta'-\beta)x} dx = \frac{1}{\beta'(\beta'-\beta)}\|\lambda\|_{\beta'}^2,\end{aligned}$$

proving the assertion. □

5. Invariant foliations for the HJMM equation

We shall now investigate invariant foliations for the HJMM equation (1.1) by working on the space of forward curves from the previous section.

Let $0 < \beta < \beta'$ be arbitrary real numbers and let $\sigma : H_\beta \to H_\beta$ be given.

5.1. **Assumption.** *We assume that $\sigma \in C^1(H_\beta)$ with $\sigma(H_\beta) \subset H^0_{\beta'}$ and that there exist $L, M > 0$ such that*

$$\|\sigma(h_1) - \sigma(h_2)\|_\beta \leq L\|h_1 - h_2\|_\beta \quad \text{for all } h_1, h_2 \in H_\beta,$$
$$\|\sigma(h)\|_\beta \leq M \quad \text{for all } h \in H_\beta.$$

Using the notation of the previous section, the HJM drift term (1.2) is given by

$$\alpha_{\text{HJM}} = m(\sigma, \mathcal{I}\sigma).$$

According to [11, Cor. 5.1.2] we have $\alpha_{\rm HJM}(H_\beta) \subset H_\beta^0$ and there exists a constant $K > 0$ such that

$$\|\alpha_{\rm HJM}(h_1) - \alpha_{\rm HJM}(h_2)\|_\beta \leq K \|h_1 - h_2\|_\beta \quad \text{for all } h_1, h_2 \in H_\beta.$$

Hence, for each $h_0 \in H_\beta$ there exists a unique weak solution for (1.1) with $r_0 = h_0$. Note that (1.1) is a particular example of the stochastic partial differential equation (2.1) with $A = \frac{d}{dx}$ and $\alpha = \alpha_{\rm HJM}$. Moreover, Lemmas 4.2, 4.3 yield $\alpha_{\rm HJM} \in C^1(H_\beta)$, whence all required conditions from Assumptions 2.1 are fulfilled.

Now let $(\mathcal{M}_t)_{t \geq 0}$ be a foliation generated by a finite dimensional subspace $V \subset H_\beta$. We set $d := \dim V$. In order to investigate invariance of $(\mathcal{M}_t)_{t \geq 0}$ for the HJMM equation (1.1), we directly switch to a coordinate system. Let $\psi \in C^1(\mathbb{R}_+; H)$ be a parametrization of $(\mathcal{M}_t)_{t \geq 0}$ and let $\{\lambda_1, \ldots, \lambda_d\}$ be a basis of V. Then, the set $\{\Lambda_1, \ldots, \Lambda_d\}$ is linearly independent in $C(\mathbb{R}_+)$.

5.2. **Remark.** *Let $E_1 \subset \{1, \ldots, d\}$ be an index set. We set*

$$E_2 := \{(i,j) \in E_1 \times E_1 : i < j\}.$$

Then, there are subsets $D_1 \subset E_1$ and $D_2 \subset E_2$ such that

$$B = \{\Lambda_1, \ldots, \Lambda_d\} \cup \{\Lambda_i^2 : i \in D_1\} \cup \{\Lambda_i \Lambda_j : (i,j) \in D_2\} \tag{5.1}$$

is a basis of the vector space

$$W = \langle \Lambda_1, \ldots, \Lambda_d \rangle + \langle \Lambda_i^2 : i \in E_1 \rangle + \langle \Lambda_i \Lambda_j : (i,j) \in E_2 \rangle. \tag{5.2}$$

For each $m \in E_1 \setminus D_1$ there exist unique $(c_i^m)_{i=1,\ldots,d} \subset \mathbb{R}$, $(d_i^m)_{i \in D_1} \subset \mathbb{R}$ and $(d_{ij}^m)_{(i,j) \in D_2} \subset \mathbb{R}$ such that

$$\Lambda_m^2 = \sum_{i=1}^d c_i^m \Lambda_i + \sum_{i \in D_1} d_i^m \Lambda_i^2 + \sum_{(i,j) \in D_2} d_{ij}^m \Lambda_i \Lambda_j, \tag{5.3}$$

and for each $(m,n) \in E_2 \setminus D_2$ there exist unique $(c_i^{mn})_{i=1,\ldots,d} \subset \mathbb{R}$, $(d_i^{mn})_{i \in D_1} \subset \mathbb{R}$ and $(d_{ij}^{mn})_{(i,j) \in D_2} \subset \mathbb{R}$ such that

$$\Lambda_m \Lambda_n = \sum_{i=1}^d c_i^{mn} \Lambda_i + \sum_{i \in D_1} d_i^{mn} \Lambda_i^2 + \sum_{(i,j) \in D_2} d_{ij}^{mn} \Lambda_i \Lambda_j. \tag{5.4}$$

5.3. **Theorem.** *The foliation $(\mathcal{M}_t)_{t \geq 0}$ is invariant for the HJMM equation (1.1) if and only if we have (2.20), (2.21), there exist $\mu, \gamma \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ such that*

$$\nu(\psi(t)) = \psi'(t) + \sum_{i=1}^d \mu_i(t,0) \lambda_i, \quad t \in \mathbb{R}_+ \tag{5.5}$$

and (2.23), there are $(a_i^k)_{i=1,\dots,d}^{k=1,\dots,d} \subset \mathbb{R}$, $(b_i^k)_{i\in D_1}^{k=1,\dots,d} \subset \mathbb{R}$ *and* $(b_{ij}^k)_{(i,j)\in D_2}^{k=1,\dots,d} \subset \mathbb{R}$ *such that for all* $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$ *we have*

$$(5.6) \quad -\frac{\partial}{\partial y_k}\mu_i(t,y) + \frac{1}{2}\sum_{m\in E_1\setminus D_1} c_i^m \frac{\partial}{\partial y_k}(\gamma_m(t,y)^2) + \sum_{(m,n)\in E_2\setminus D_2} c_i^{mn}\frac{\partial}{\partial y_k}(\gamma_m(t,y)\gamma_n(t,y)) = a_i^k, \quad i=1,\dots,d,\ k=1,\dots,d$$

$$(5.7) \quad \frac{1}{2}\frac{\partial}{\partial y_k}(\gamma_i(t,y)^2) + \frac{1}{2}\sum_{m\in E_1\setminus D_1} d_i^m \frac{\partial}{\partial y_k}(\gamma_m(t,y)^2) + \sum_{(m,n)\in E_2\setminus D_2} d_i^{mn}\frac{\partial}{\partial y_k}(\gamma_m(t,y)\gamma_n(t,y)) = b_i^k, \quad i\in D_1,\ k=1,\dots,d$$

$$(5.8) \quad \frac{\partial}{\partial y_k}(\gamma_i(t,y)\gamma_j(t,y)) + \frac{1}{2}\sum_{m\in E_1\setminus D_1} d_{ij}^m \frac{\partial}{\partial y_k}(\gamma_m(t,y)^2) + \sum_{(m,n)\in E_2\setminus D_2} d_{ij}^{mn}\frac{\partial}{\partial y_k}(\gamma_m(t,y)\gamma_n(t,y)) = b_{ij}^k, \quad (i,j)\in D_2,\ k=1,\dots,d$$

where $E_1 \subset \{1,\dots,d\}$ *is chosen such that* $E_1 \supset \{i=1,\dots,d : \gamma_i \not\equiv 0\}$ *and the further quantities are chosen as in Remark 5.2, and we have the Ricatti equations*

$$(5.9) \quad \frac{d}{dx}\Lambda_k + \sum_{i=1}^d a_i^k\Lambda_i + \sum_{i\in D_1} b_i^k\Lambda_i^2 + \sum_{(i,j)\in D_2} b_{ij}^k\Lambda_i\Lambda_j = \lambda_k(0), \quad k=1,\dots,d.$$

Proof. "⇒" Suppose $(\mathcal{M}_t)_{t\geq 0}$ is an invariant foliation for (1.1). According to Corollary 2.13 we have (2.20)–(2.23). Relation (5.5) follows by setting $y=0$ in (2.22). Inserting (2.23) into (2.22) we get, by taking into account the HJM drift condition (1.2),

$$\frac{d}{dx}\psi(t) + \sum_{i=1}^d y_i\frac{d}{dx}\lambda_i + \frac{1}{2}\frac{d}{dx}\left(\sum_{i=1}^d \gamma_i(t,y)\Lambda_i\right)^2 = \psi'(t) + \sum_{i=1}^d \mu_i(t,y)\lambda_i$$

for all $(t,y)\in\mathbb{R}_+\times\mathbb{R}^d$. Differentiating with respect to y_k we obtain

$$\frac{d}{dx}\lambda_k + \frac{d}{dx}\left(\left(\sum_{i=1}^d \gamma_i(t,y)\Lambda_i\right)\left(\sum_{i=1}^d \frac{\partial}{\partial y_k}\gamma_i(t,y)\Lambda_i\right)\right) = \sum_{i=1}^d \frac{\partial}{\partial y_k}\mu_i(t,y)\lambda_i, \quad k=1,\dots,d$$

for all $(t,y)\in\mathbb{R}_+\times\mathbb{R}^d$. Integrating yields

$$\lambda_k - \sum_{i=1}^d \frac{\partial}{\partial y_k}\mu_i(t,y)\Lambda_i + \sum_{i,j=1}^d \gamma_i(t,y)\frac{\partial}{\partial y_k}\gamma_j(t,y)\Lambda_i\Lambda_j = \lambda_k(0), \quad k=1,\dots,d$$

for all $(t,y)\in\mathbb{R}_+\times\mathbb{R}^d$. Noting that $E_1\supset\{i=1,\dots,d:\gamma_i\not\equiv 0\}$, we can express this equation as

$$\lambda_k - \sum_{i=1}^d \frac{\partial}{\partial y_k}\mu_i(t,y)\Lambda_i + \frac{1}{2}\sum_{i\in E_1}\frac{\partial}{\partial y_k}(\gamma_i(t,y)^2)\Lambda_i^2 + \sum_{(i,j)\in E_2}\frac{\partial}{\partial y_k}(\gamma_i(t,y)\gamma_j(t,y))\Lambda_i\Lambda_j = \lambda_k(0), \quad k=1,\dots,d$$

for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$. Introducing the functions $f_i : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, d$ and $g_i : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $i \in D_1$ as well as $g_{ij} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $(i,j) \in D_2$ by

$$\begin{aligned} f_i(t,y) &:= -\mu_i(t,y) + \frac{1}{2} \sum_{m \in E_1 \setminus D_1} c_i^m \gamma_m(t,y)^2 + \sum_{(m,n) \in E_2 \setminus D_2} c_i^{mn} \gamma_m(t,y)\gamma_n(t,y), \\ g_i(t,y) &:= \frac{1}{2}\gamma_i(t,y)^2 + \frac{1}{2} \sum_{m \in E_1 \setminus D_1} d_i^m \gamma_m(t,y)^2 + \sum_{(m,n) \in E_2 \setminus D_2} d_i^{mn} \gamma_m(t,y)\gamma_n(t,y), \\ g_{ij}(t,y) &:= \gamma_i(t,y)\gamma_j(t,y) + \frac{1}{2} \sum_{m \in E_1 \setminus D_1} d_{ij}^m \gamma_m(t,y)^2 \\ &\quad + \sum_{(m,n) \in E_2 \setminus D_2} d_{ij}^{mn} \gamma_m(t,y)\gamma_n(t,y), \end{aligned}$$

we obtain, by taking into account (5.3) and (5.4),

$$\begin{aligned} &\sum_{i=1}^d \frac{\partial}{\partial y_k} f_i(t,y)\Lambda_i + \sum_{i \in D_1} \frac{\partial}{\partial y_k} g_i(t,y)\Lambda_i^2 + \sum_{(i,j) \in D_2} \frac{\partial}{\partial y_k} g_{ij}(t,y)\Lambda_i\Lambda_j \\ &= -(\lambda_k - \lambda_k(0)), \quad k = 1, \dots, d \end{aligned}$$

for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$. Since B defined in (5.1) is a basis of the vector space W in (5.2), we deduce (5.6), (5.7), (5.8) and the Ricatti equations (5.9).

"$\Leftarrow$": Relations (1.2), (2.23), (5.3), (5.4) yield

$$\begin{aligned} &\alpha_{\text{HJM}}\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i\bigg) = \frac{1}{2}\frac{d}{dx}\bigg(\sum_{i,j=1}^d \gamma_i(t,y)\gamma_j(t,y)\Lambda_i\Lambda_j\bigg) \\ &= \frac{1}{2}\frac{d}{dx}\bigg(\sum_{i \in E_1} \gamma_i(t,y)^2\Lambda_i^2 + 2\sum_{(i,j) \in E_2} \gamma_i(t,y)\gamma_j(t,y)\Lambda_i\Lambda_j\bigg) \\ &= \frac{d}{dx}\bigg(\sum_{i=1}^d (\mu_i(t,y) + f_i(t,y))\Lambda_i + \sum_{i \in D_1} g_i(t,y)\Lambda_i^2 + \sum_{(i,j) \in D_2} g_{ij}(t,y)\Lambda_i\Lambda_j\bigg) \end{aligned} \tag{5.10}$$

for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$. In particular, by setting $y = 0$, we have

$$\begin{aligned} &\alpha_{\text{HJM}}(\psi(t)) \\ &= \frac{d}{dx}\bigg(\sum_{i=1}^d (\mu_i(t,0) + f_i(t,0))\Lambda_i + \sum_{i \in D_1} g_i(t,0)\Lambda_i^2 + \sum_{(i,j) \in D_2} g_{ij}(t,0)\Lambda_i\Lambda_j\bigg) \end{aligned} \tag{5.11}$$

for all $t \in \mathbb{R}_+$. Relations (5.10), (5.6), (5.7), (5.8), (5.11) and the Ricatti equations (5.9) give us

$$\begin{aligned}
\alpha_{\mathrm{HJM}}\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i\bigg) &= \frac{d}{dx}\bigg(\sum_{i=1}^d \bigg(\mu_i(t,y) + f_i(t,0) + \sum_{k=1}^d a_i^k y_k \bigg) \Lambda_i \\
&\quad + \sum_{i \in D_1} \bigg(g_i(t,0) + \sum_{k=1}^d b_i^k y_k \bigg) \Lambda_i^2 + \sum_{(i,j) \in D_2} \bigg(g_{ij}(t,0) + \sum_{k=1}^d b_{ij}^k y_k \bigg) \Lambda_i \Lambda_j \bigg) \\
&= \frac{d}{dx}\bigg(\sum_{i=1}^d (\mu_i(t,0) + f_i(t,0)) \Lambda_i + \sum_{i \in D_1} g_i(t,0) \Lambda_i^2 + \sum_{(i,j) \in D_2} g_{ij}(t,0) \Lambda_i \Lambda_j \bigg) \\
&\quad + \frac{d}{dx} \sum_{i=1}^d (\mu_i(t,y) - \mu_i(t,0)) \Lambda_i \\
&\quad + \frac{d}{dx} \sum_{k=1}^d y_k \bigg(\sum_{i=1}^d a_i^k \Lambda_i + \sum_{i \in D_1} b_i^k \Lambda_i^2 + \sum_{(i,j) \in D_2} b_{ij}^k \Lambda_i \Lambda_j \bigg) \\
&= \alpha_{\mathrm{HJM}}(\psi(t)) + \sum_{i=1}^d (\mu_i(t,y) - \mu_i(t,0)) \lambda_i - \sum_{k=1}^d y_k \frac{d}{dx} \lambda_k
\end{aligned}$$

for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$. We conclude, by furthermore incorporating (5.5),

$$\begin{aligned}
\nu\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i\bigg) &= \frac{d}{dx}\psi(t) + \sum_{i=1}^d y_i \frac{d}{dx} \lambda_i + \alpha_{\mathrm{HJM}}\bigg(\psi(t) + \sum_{i=1}^d y_i \lambda_i\bigg) \\
&= \nu(\psi(t)) + \sum_{i=1}^d (\mu_i(t,y) - \mu_i(t,0)) \lambda_i = \psi'(t) + \sum_{i=1}^d \mu_i(t,y) \lambda_i
\end{aligned}$$

for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d$, showing (2.22). According to Corollary 2.13, the foliation $(\mathcal{M}_t)_{t \geq 0}$ is an invariant for (1.1). □

5.4. **Remark.** *Note that in particular the system (5.9) of Ricatti equations is useful in order to gain knowledge about the existence of an affine realization. We will exemplify Theorem 5.3 in the upcoming section.*

6. Affine realizations for the HJMM equation

We shall now demonstrate Theorem 5.3 in order to characterize volatility structures for which the HJMM equation (1.1) admits an affine realization. We start with general volatilities, and will obtain results for particular volatility structures as corollaries.

6.1. **General volatility.** In this subsection, we assume that the volatility σ in the HJMM equation (1.1) is of the form

$$\sigma(h) = \sum_{i=1}^p \Phi_i(h) \lambda_i, \quad h \in H_\beta \tag{6.1}$$

where $p \in \mathbb{N}$ denotes a positive integer, $\Phi_1, \ldots, \Phi_p : H_\beta \to \mathbb{R}$ are functionals and $\lambda_1, \ldots, \lambda_p \in H_{\beta'}^0$ are linearly independent. We assume that $\Phi_i \in C^2(H_\beta; \mathbb{R})$ for $i = 1, \ldots, p$ and that there exist $L, M > 0$ such that for all $i = 1, \ldots, p$ we have

$$\begin{aligned}
|\Phi_i(h_1) - \Phi_i(h_2)| &\leq L \|h_1 - h_2\|_\beta \quad \text{for all } h_1, h_2 \in H_\beta, \\
|\Phi_i(h)| &\leq M \quad \text{for all } h \in H_\beta.
\end{aligned}$$

Then, Assumption 5.1 is fulfilled.

Note that, in view of Lemma 3.2, this is the most general volatility, which we can have for the HJMM equation (1.1) with an affine realization. The corresponding HJM drift term (1.2) is given by

$$\alpha_{\mathrm{HJM}}(h) = \bigg(\sum_{i=1}^p \Phi_i(h)\lambda_i \bigg) \bigg(\sum_{i=1}^p \Phi_i(h)\Lambda_i \bigg), \quad h \in H_\beta. \tag{6.2}$$

6.1. **Proposition.** *Suppose there exist $h_1, \dots, h_p \in H_\beta$ such that $\sigma(h_1), \dots, \sigma(h_p)$ are linearly independent, and $h_0 \in \mathcal{D}(\frac{d}{dx})$ such that one of the following conditions is satisfied:*

- *We have*

$$D(\Phi_i \Phi_j)(h_0)\lambda_k = 0, \quad i,j,k = 1, \dots, p. \tag{6.3}$$

- *There exist $k, l \in \{1, \dots, p\}$ such that the functions*

$$h \mapsto D^2(\Phi_i \Phi_j)(h)(\lambda_k, \lambda_l), \quad h \in h_0 + \langle \lambda_1, \dots, \lambda_p \rangle \tag{6.4}$$

  are linearly independent for $1 \leq i \leq j \leq p$.

If the HJMM equation (1.1) has an affine realization, then $\lambda_1, \dots, \lambda_p$ are quasi-exponential.

Proof. Let $V \subset H_\beta$ be a finite dimensional subspace generating the affine realization and set $d := \dim V$. Lemma 3.2 yields that $\sigma(h) \in V$ for all $h \in H_\beta$. Since $\sigma(h_1), \dots, \sigma(h_p)$ are linearly independent, relation (6.1) yields that $\lambda_1, \dots, \lambda_p \in V$. Choose $\lambda_{p+1}, \dots, \lambda_d \in V$ such that $\{\lambda_1, \dots, \lambda_d\}$ is a basis of V. Let $h_0 \in \mathcal{D}(\frac{d}{dx})$ be such that one of the conditions above is satisfied. Now we apply Theorem 5.3 to the invariant foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$, implying the existence of $\gamma \in C^2(\mathbb{R}^d)$ such that

$$\gamma_i(y) = \Phi_i\bigg(h_0 + \sum_{i=1}^d y_i \lambda_i \bigg), \quad i = 1, \dots, p$$
$$\gamma_i(y) = 0, \quad i = p+1, \dots, d$$

whence we can choose $E_1 = \{1, \dots, p\}$.

If (6.3) is satisfied, then (5.7), (5.8) give us $b_i^k = 0$ for all $i \in D_1$, $k = 1, \dots, p$ and $b_{ij}^k = 0$ for all $(i,j) \in D_2$, $k = 1, \dots, p$. Consequently, the Ricatti equations (5.9) show that $\lambda_1, \dots, \lambda_p$ are quasi-exponential.

If there exist $k, l \in \{1, \dots, p\}$ such that the functions (6.4) are linearly independent for $1 \leq i \leq j \leq p$, then we claim that $D_1 = D_2 = \emptyset$, which, in view of the Ricatti equations (5.9), implies that $\lambda_1, \dots, \lambda_p$ are quasi-exponential. Suppose, on the contrary, that $D_1 \neq \emptyset$ or $D_2 \neq \emptyset$.

If $D_1 \neq \emptyset$, choose $i \in D_1$ and differentiate (5.7) with respect to y_l, which yields

$$\begin{aligned} &\frac{1}{2} D^2 \Phi_i^2(h)(\lambda_k, \lambda_l) + \frac{1}{2} \sum_{m \in E_1 \setminus D_1} d_i^m D^2 \Phi_m^2(h)(\lambda_k, \lambda_l) \\ &+ \sum_{(m,n) \in E_2 \setminus D_2} d_i^{mn} D^2(\Phi_m \Phi_n)(h)(\lambda_k, \lambda_l) = 0 \end{aligned}$$

for all $h \in h_0 + \langle \lambda_1, \dots, \lambda_p \rangle$. This contradicts the linear independence of (6.4) for $1 \leq i \leq j \leq p$.

Analogously, if $D_2 \neq \emptyset$, choosing $(i,j) \in D_2$ and differentiating (5.8) with respect to y_l yields a contradiction to the linear independence of (6.4) for $1 \leq i \leq j \leq p$. □

6.2. **Proposition.** *If $\lambda_1, \dots, \lambda_p$ are quasi-exponential, then the HJMM equation (1.1) has an affine realization.*

Proof. Using Lemma 3.6 there exists a finite dimensional subspace $W \subset \mathcal{D}(\frac{d}{dx})$ with $\lambda_1, \dots, \lambda_p \in W$ and

$$\frac{d}{dx} w \in W \quad \text{for all } w \in W. \tag{6.5}$$

Since $S_t H^0_{\beta'} \subset H^0_{\beta'}$ for all $t \geq 0$, we have $W \subset H^0_{\beta'}$. Set $q := \dim W$. There exist $\lambda_{p+1}, \dots, \lambda_q \in W$ such that $\{\lambda_1, \dots, \lambda_q\}$ is a basis of W. We define the subspace

$$V := W + \langle \lambda_i \Lambda_j : i, j = 1, \dots, q \rangle.$$

Note that $V \subset \mathcal{D}(\frac{d}{dx})$ by Lemmas 4.2, 4.3. Set $d := \dim V$ and choose $\lambda_{q+1}, \dots, \lambda_d \in V$ such that $\{\lambda_1, \dots, \lambda_d\}$ is a basis of V. Relation (6.5) implies

$$\frac{d}{dx} \Lambda_i \in \langle 1, \Lambda_1, \dots, \Lambda_q \rangle, \quad i = 1, \dots, q. \tag{6.6}$$

By (6.6) and (6.5) we have

$$\frac{d}{dx}(\lambda_i \Lambda_j) = \lambda_i \frac{d}{dx} \Lambda_j + \Lambda_j \frac{d}{dx} \lambda_i \in V, \quad i, j = 1, \dots, q$$

whence we have

$$\frac{d}{dx} v \in V \quad \text{for all } v \in V. \tag{6.7}$$

Let $h_0 \in \mathcal{D}(\frac{d}{dx})$ be arbitrary. We define the map $\psi \in C^1(\mathbb{R}_+; H_\beta)$,

$$\psi(t) := S_t h_0 \tag{6.8}$$

the map $\gamma \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$,

$$\gamma_i(t, y) := \begin{cases} \Phi_i(\psi(t) + \sum_{j=1}^d y_j \lambda_j), & i = 1, \dots, p \\ 0, & i = p+1, \dots, d. \end{cases} \tag{6.9}$$

and $\mu \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d)$ as

(6.10)
$$\mu_i(t, y) := \begin{cases} \gamma_k(t, y) \gamma_l(t, y) + \sum_{j=1}^d a_{ji} y_j & \text{if } i \in \{q+1, \dots, d\} \text{ and } \lambda_i = \lambda_k \Lambda_l \\ & \text{for some } k, l \in \{1, \dots, p\}, \\ \sum_{j=1}^d a_{ji} y_j, & \text{otherwise,} \end{cases}$$

where, due to (6.7), the $(a_{ij})_{i,j=1,\dots,d} \subset \mathbb{R}$ are chosen such that

$$\frac{d}{dx} \lambda_i = \sum_{j=1}^d a_{ij} \lambda_j, \quad i = 1, \dots, d.$$

Then, conditions (2.20)–(2.23) are fulfilled, and therefore, by Corollary 2.13, the foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ generated by $\langle \lambda_1, \dots, \lambda_d \rangle$ with parametrization ψ is invariant for the HJMM equation (1.1). □

6.3. **Remark.** *Note that the proof of Proposition 6.2 simultaneously provides the construction of the affine realization. For $h_0 \in \mathcal{D}(\frac{d}{dx})$ the invariant foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ is generated by $\langle \lambda_1, \dots, \lambda_d \rangle$ and has the parametrization ψ defined in (6.8). For $h \in \mathcal{M}_{t_0}^{h_0}$ with some $t_0 \in \mathbb{R}_+$ the strong solution $(r_t)_{t \geq 0}$ for (1.1) with $r_0 = h$ is given by (2.27), where the maps $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ for the state process (2.26) are defined in (6.9), (6.10). We refer to* [5, Prop. 5.1, Prop. 6.1] *for similar results.*

6.4. **Remark.** *Combining Proposition 3.3 and relations (6.2), (6.8), the singular set Σ is given by the $(d+1)$-dimensional linear space*

$$\Sigma = \langle 1, \Lambda_1, \dots, \Lambda_d \rangle,$$

and for each $h_0 \in \mathcal{D}(\frac{d}{dx})$ we have (3.4), (3.5), where

$$t_0 := \inf\{t \geq 0 : S_t h_0 \in \Sigma\} \in [0, \infty]$$

and where $(r_t)_{t \geq 0}$ denotes the strong solution for (1.1) with $r_0 = h_0$.

6.5. **Remark.** *Note that the conditions in Proposition 6.1 are singular events, because the respective conditions only have to be satisfied for one single point h_0. Hence, Propositions 6.1, 6.2 yield that, apart from degenerate examples like the CIR model, the existence of an affine realization is essentially equivalent to the condition that $\lambda_1, \dots, \lambda_p$ are quasi-exponential (which means that all quadratic terms in the system (5.9) of Ricatti equations disappear). This also supplements* [6, Prop. 6.4], *which provides the sufficient implication.*

6.2. **Constant direction volatility.** In this subsection, we study the existence of affine realizations for the HJMM equation (1.1) with constant direction volatility, that is, we assume that the volatility σ in the HJMM equation (1.1) is of the form

$$\sigma(h) = \Phi(h)\lambda, \quad h \in H_\beta \tag{6.11}$$

where $\Phi : H_\beta \to \mathbb{R}$ is a functional and $\lambda \in H^0_{\beta'}$ with $\lambda \neq 0$. We assume that $\Phi \in C^2(H_\beta; \mathbb{R})$ and that there exist $L, M > 0$ such that

$$|\Phi(h_1) - \Phi(h_2)| \leq L \|h_1 - h_2\|_\beta \quad \text{for all } h_1, h_2 \in H_\beta,$$
$$|\Phi(h)| \leq M \quad \text{for all } h \in H_\beta.$$

Then, Assumption 5.1 is fulfilled.

6.6. **Corollary.** *Suppose $\Phi \not\equiv 0$. If the HJMM equation (1.1) has an affine realization, then λ is quasi-exponential or we have*

$$D^2\Phi^2(h)(\lambda, \lambda) = 0, \quad h \in \mathcal{D}\bigg(\frac{d}{dx}\bigg) \tag{6.12}$$

$$D\Phi^2(h)\lambda \neq 0, \quad h \in \mathcal{D}\bigg(\frac{d}{dx}\bigg). \tag{6.13}$$

Proof. This is an immediate consequence of Proposition 6.1. □

6.7. **Remark.** *Conditions (6.12), (6.13) mean that at each forward curve $h \in H_\beta$ the functional Φ^2 is affine, but not constant, in direction λ, which is the typical feature for CIR type models.*

6.8. **Corollary.** *If λ is quasi-exponential, then the HJMM equation (1.1) has an affine realization.*

Proof. This is a direct consequence of Proposition 6.2. □

6.9. **Remark.** *Suppose we have $\Phi \not\equiv 0$ and there exists $h_0 \in \mathcal{D}(\frac{d}{dx})$ such that*

$$D^2\Phi^2(h_0)(\lambda, \lambda) \neq 0 \quad \textit{or} \quad D\Phi^2(h_0)\lambda = 0.$$

Then, by Corollaries 6.6, 6.8, the HJMM equation (1.1) has an affine realization if and only if λ is quasi-exponential. Hence, we have relaxed the assumptions from [6, Prop. 6.1], *where it is assumed that $\Phi(h) \neq 0$ for all $h \in H_\beta$ and $D^2\Phi^2(h)(\lambda, \lambda) \neq 0$ for all $h \in H_\beta$.*

6.3. **Constant volatility.** In this subsection, we study the existence of affine realizations for the HJMM equation (1.1) with constant volatility, i.e., we have

$$\sigma \equiv \lambda$$

with $\lambda \in H^0_{\beta'}$, $\lambda \neq 0$. Then, Assumption 5.1 is fulfilled.

6.10. **Corollary.** *The HJMM equation (1.1) has an affine realization if and only if λ is quasi-exponential.*

Proof. The assertion is a direct consequence of Corollaries 6.6, 6.8, because σ is of the form (6.11) with $\Phi \equiv 1$. □

6.11. **Remark.** *If λ is quasi-exponential, we even obtain a d-dimensional affine realization, where $d := \dim\langle (\frac{d}{dx})^n \lambda : n \in \mathbb{N}_0 \rangle$. For $h_0 \in \mathcal{D}(\frac{d}{dx})$ the invariant foliation $(\mathcal{M}_t^{h_0})_{t \geq 0}$ is generated by $\langle \lambda_1, \dots, \lambda_d \rangle$ with*

$$\lambda_i = \left(\frac{d}{dx} \right)^{i-1} \lambda, \quad i = 1, \dots, d$$

and has the parametrization

$$\psi(t) = -\frac{1}{2}\Lambda^2 + S_t\bigg(h_0 + \frac{1}{2}\Lambda^2 \bigg), \quad t \in \mathbb{R}_+$$

which can be shown by using Corollary 2.13 (cf. [5, Prop. 4.1]*). Using Proposition 3.3, the singular set Σ is given by the $(d+1)$-dimensional affine space*

$$\Sigma = -\frac{1}{2}\Lambda^2 + \langle 1, \Lambda_1, \dots, \Lambda_d \rangle,$$

and for each $h_0 \in \mathcal{D}(\frac{d}{dx})$ we have (3.4), (3.5), where

$$t_0 := \inf \bigg\{ t \geq 0 : S_t\bigg(h_0 + \frac{1}{2}\Lambda^2 \bigg) \in \langle 1, \Lambda_1, \dots, \Lambda_d \rangle \bigg\} \in [0, \infty]$$

and where $(r_t)_{t \geq 0}$ denotes the strong solution for (1.1) with $r_0 = h_0$.

6.12. **Remark.** *Not surprising, the statement of Corollary 6.10 coincides with that of* [6, Prop. 5.1].

6.4. **Short rate realizations.** Using our previous results, we can easily show that only the following three types of affine short rate realizations exist:

- The Ho-Lee model,
- The Hull-White extension of the Vasicek model,
- The Hull-White extension of the Cox-Ingersoll-Ross model,

cf., e.g., [16, 6, 13]. We omit further details here.

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