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A NOTE ON STOCHASTIC INTEGRALS AS L^2-CURVES

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Abstract. In a work of van Gaans [8] stochastic integrals are regarded as L^2-curves. In [5] we have shown the connection to the usual Itô-integral for càdlàg-integrands. The goal of this note is to complete this result and to provide the full connection to the Itô-integral. We also sketch an application to stochastic partial differential equations.

Key Words: Stochastic integrals, L^2-curves, connection to the Itô-integral, stochastic partial differential equations.

60H05, 60H15

1. Introduction

In the paper [5] we have established an existence and uniqueness result for stochastic partial differential equations, driven by Lévy processes, by applying a result from van Gaans [8, Thm. 4.1]. In [8] stochastic integrals are regarded as L^2-curves. It was therefore necessary to establish the connection to the usual Itô-integral (developed, e.g., in [9] or [13]) for càdlàg-integrands, which we have provided in [5, Appendix B].

The goal of the present note is to complete this result and to provide the full connection to the Itô-integral. More precisely, we will show that the space of adapted L^2-curves is embedded into the space of Itô-integrable processes (see Proposition 2.5 below), and that the corresponding Itô-integral is a càdlàg-version of the stochastic integral in the sense of [8] (see Proposition 2.9 below).

This is the content of Section 2. Afterwards, we outline an application to stochastic partial differential equations in Section 3.

2. Stochastic integrals as L^2-curves

Throughout this text, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Furthermore, let $(H, \| \cdot \|)$ denote a separable Hilbert space.

For any $T \in \mathbb{R}_+$ the space $C[0,T] := C([0,T]; L^2(\Omega; H))$ of all curves from $[0,T]$ into $L^2(\Omega; H)$ is a Banach space with respect to the norm

$$\|r\|_T := \sup_{t \in [0,T]} \|r_t\|_{L^2(\Omega;H)} = \sqrt{\sup_{t \in [0,T]} \mathbb{E}[\|r_t\|^2]}.$$

The subspace $C_{\text{ad}}[0,T]$ consisting of all adapted processes from $C[0,T]$ is closed with respect to this norm. Note that, by the completeness of the filtration $(\mathcal{F}_t)_{t \geq 0}$, adaptedness is independent of the choice of the representative.

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2.1. **Stochastic integral with respect to a Lévy martingale.** Let M be a real-valued, square-integrable Lévy martingale. We recall how in this case the stochastic integral (G-)$(\Phi \cdot M)$ in the sense of van Gaans [8, Sec. 3] is defined for $\Phi \in C_{\mathrm{ad}}[0,T]$.

2.1. **Lemma.** [8, Prop. 3.2.1] *Let $\Phi \in C_{\mathrm{ad}}[0,T]$ be arbitrary. For each $t \in [0,T]$ there exists a unique random variable $Y_t \in L^2(\Omega)$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\mathbb{E}\left[\left\| Y_t - \sum_{i=0}^{n-1} \Phi_{t_i}(M_{t_{i+1}} - M_{t_i}) \right\|^2\right] < \varepsilon \tag{2.1}$$

for every partition $0 = t_0 < t_1 < \ldots < t_n = t$ with $\sup_{i=0,\ldots,n-1} |t_{i+1} - t_i| < \delta$.

2.2. **Definition.** [8] *Let $\Phi \in C_{\mathrm{ad}}[0,T]$ be arbitrary. Then the stochastic integral $Y =$ (G-)$(\Phi \cdot M)$ is the stochastic process $Y = (Y_t)_{t \in [0,T]}$ where every Y_t is the unique element from $L^2(\Omega)$ such that (2.1) is valid.*

We observe that the integrand Φ as well as the stochastic integral (G-)$(\Phi \cdot M)$ are only determined up to a version. In particular, it is not clear if the integral process has a càdlàg-version.

2.3. **Lemma.** [8, Thm. 3.3.2] *For each $\Phi \in C_{\mathrm{ad}}[0,T]$ we have*

$$(\text{G-})(\Phi \cdot M) \in C_{\mathrm{ad}}[0,T].$$

We are now interested in finding the connection between the stochastic integral (G-)$(\Phi \cdot M)$ and the usual Itô-integral (developed, e.g., in [9, 13, 2] for the finite dimensional case and in [3, 12] for the infinite dimensional case). We use the abbreviation

$$L^2(\mathcal{P}_T) := L^2(\Omega \times [0,T], \mathcal{P}_T, \mathbb{P} \otimes \lambda; H),$$

where $\mathcal{P}_T$ denotes the predictable σ-algebra on $\Omega \times [0,T]$ and λ the Lebesgue measure. Since for any square-integrable Lévy martingale M the predictable quadratic covariation $\langle M, M \rangle$ is linear, $L^2(\mathcal{P}_T)$ is the space of all L^2-processes Φ, for which the Itô-integral $\Phi \cdot M$ exists, independent of the choice of M.

2.4. **Lemma.** *For each $\Phi \in C_{\mathrm{ad}}[0,T]$ there exists a predictable version ${}^p\Phi \in L^2(\mathcal{P}_T)$ of Φ.*

Proof. By [3, Prop. 3.6.ii] there exists a predictable version ${}^p\Phi$ of Φ. Since $\Phi \in C_{\mathrm{ad}}[0,T]$, we also have

$$\int_0^T \mathbb{E}[\|{}^p\Phi_t\|^2] dt = \int_0^T \mathbb{E}[\|\Phi_t\|^2] dt \le T \sup_{t \in [0,T]} \mathbb{E}[\|\Phi_t\|^2] < \infty,$$

that is ${}^p\Phi \in L^2(\mathcal{P}_T)$. □

2.5. **Proposition.** *The map $\Phi \mapsto {}^p\Phi$ defines an embedding from $C_{\mathrm{ad}}[0,T]$ into $L^2(\mathcal{P}_T)$.*

Proof. For two predictable versions $\Phi^1, \Phi^2 \in L^2(\mathcal{P}_T)$ of Φ we have

$$\int_0^T \mathbb{E}[\|\Phi_t^1 - \Phi_t^2\|^2] dt = 0,$$

whence $\Phi^1 = \Phi^2$ in $L^2(\mathcal{P}_T)$. Therefore, the map $\Phi \mapsto {}^p\Phi$ is well-defined. The linearity of $\Phi \mapsto {}^p\Phi$ is immediately checked, and the estimate

$$\int_0^T \mathbb{E}[\|{}^p\Phi_t\|^2] dt = \int_0^T \mathbb{E}[\|\Phi_t\|^2] dt \le T \sup_{t \in [0,T]} \mathbb{E}[\|\Phi_t\|^2], \quad \Phi \in C_{\mathrm{ad}}[0,T]$$

proves the continuity of $\Phi \mapsto {}^p\Phi$. For $\Phi \in C_{\mathrm{ad}}[0,T]$ with ${}^p\Phi = 0$ in $L^2(\mathcal{P}_T)$ we have

$$\int_0^T \mathbb{E}[\|\Phi_t\|^2]dt = \int_0^T \mathbb{E}[\|{}^p\Phi_t\|^2]dt = 0.$$

Since $\Phi \in C_{\mathrm{ad}}[0,T]$, the map $t \mapsto \mathbb{E}[\|\Phi_t\|^2]$ is continuous, which implies $\Phi = 0$ in $C_{\mathrm{ad}}[0,T]$, showing that $\Phi \mapsto {}^p\Phi$ is injective. □

The notation ${}^p\Phi$ reminds of the *predictable projection* of a process Φ, which we shall briefly recall. In the real-valued case one defines, for every $\overline{\mathbb{R}}$-valued and $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable process Φ the predictable projection ${}^\pi\Phi$ of Φ, according to [9, Thm. I.2.28], as the (up to an evanescent set) unique $(-\infty,\infty]$-valued process satisfying the following two conditions:

(1) It is predictable;
(2) $({}^\pi\Phi)_\tau = \mathbb{E}[\Phi_\tau \,|\, \mathcal{F}_{\tau-}]$ on $\{\tau \leq T\}$ for all predictable times τ.

Note that for every predictable process Φ we have ${}^\pi\Phi = \Phi$.

We transfer this definition to any H-valued process $\tilde{\Phi}$, which is an $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable representative of a process $\Phi \in C_{\mathrm{ad}}[0,T]$ by using the notion of conditional expectation from [3, Sec. 1.3]. Then, the second property of the predictable projection ensures that ${}^\pi\tilde{\Phi}$ is finite, i.e. H-valued.

We obtain the following relation between the embedding ${}^p\Phi$ and the predictable projection ${}^\pi\tilde{\Phi}$:

2.6. **Lemma.** *For each $\Phi \in C_{\mathrm{ad}}[0,T]$ and every $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable representative $\tilde{\Phi}$ we have*

$$ {}^\pi\tilde{\Phi} = {}^p\Phi \quad \text{in } L^2(\mathcal{P}_T).$$

Proof. For each $t \in [0,T]$ the identities

$$ {}^\pi\tilde{\Phi}_t = \mathbb{E}[\tilde{\Phi}_t \,|\, \mathcal{F}_{t-}] = \mathbb{E}[\Phi_t \,|\, \mathcal{F}_{t-}] = \mathbb{E}[{}^p\Phi_t \,|\, \mathcal{F}_{t-}] = {}^p\Phi_t \quad \mathbb{P}\text{–a.s.}$$

are valid, which gives us

$$\int_0^T \mathbb{E}[\|{}^\pi\tilde{\Phi}_t - {}^p\Phi_t\|^2]dt = 0,$$

proving the claimed result. □

2.7. **Lemma.** *If $\Phi \in C_{\mathrm{ad}}[0,T]$ has a càdlàg-version, then we have*

$$ {}^p\Phi = \Phi_- \quad \text{in } L^2(\mathcal{P}_T).$$

Proof. The process Φ_- is predictable and we have

$$\mathbb{E}\bigg[\int_0^T \|\Phi_t - \Phi_{t-}\|^2 dt\bigg] = \mathbb{E}\bigg[\int_0^T \|\Delta\Phi_t\|^2 dt\bigg] = 0,$$

because $\mathcal{N}_\omega = \{t \in [0,T] : \Delta\Phi_t(\omega) \neq 0\}$ is countable for all $\omega \in \Omega$. □

2.8. **Proposition.** *For each $\Phi \in C_{\mathrm{ad}}[0,T]$ we have*

$$(\text{G-})(\Phi \cdot M) = {}^p\Phi \cdot M \quad \text{in } C_{\mathrm{ad}}[0,T].$$

In particular, (G-)$(\Phi \cdot M)$ *has a càdlàg-version.*

Proof. Let $t \in [0,T]$ and $\epsilon > 0$ be arbitrary. Since $\Phi \in C_{\mathrm{ad}}[0,T]$, it is uniformly continuous on the compact interval $[0,t]$, and thus there exists $\delta > 0$ such that

$$\mathbb{E}[\|\Phi_u - \Phi_v\|^2] < \frac{\epsilon}{\langle M, M\rangle_t} \tag{2.2}$$

for all $u, v \in [0, t]$ with $|u - v| < \delta$. Let $\mathcal{Z} = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be an arbitrary decomposition with $\sup_{i=0,\ldots,n-1} |t_{i+1} - t_i| < \delta$. Defining

$$\Phi^{\mathcal{Z}} := \Phi_0 \mathbb{1}_{[0]} + \sum_{i=0}^{n-1} \Phi_{t_i} \mathbb{1}_{(t_i, t_{i+1}]},$$

we obtain, by using the Itô-isometry and (2.2),

$$\mathbb{E}\left[\left\| ({}^p\Phi \cdot M)_t - \sum_{i=0}^{n-1} \Phi_{t_i} (M_{t_{i+1}} - M_{t_i}) \right\|^2\right] = \mathbb{E}\left[\left\| \int_0^t ({}^p\Phi_s - \Phi_s^{\mathcal{Z}}) dM_s \right\|^2\right]$$

$$= \mathbb{E}\left[\int_0^t \|{}^p\Phi_s - \Phi_s^{\mathcal{Z}}\|^2 d\langle M, M\rangle_s\right] = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[\|\Phi_s - \Phi_{t_i}\|^2] d\langle M, M\rangle_s < \epsilon,$$

establishing that ${}^p\Phi \cdot M$ is a version of (G-)$(\Phi \cdot M)$. □

2.2. **Stochastic integral with respect to Lebesgue measure.** In an analogous fashion, we introduce the stochastic integral (G-)$(\Phi \cdot \lambda)$ with respect to the Lebesgue measure λ (cf. [8, Lemma 3.6]). By similar arguments as in the previous subsection, we obtain the same relation between this stochastic integral (G-)$(\Phi \cdot \lambda)$ and the usual Bochner integral $\Phi \cdot \lambda$.

2.3. **Stochastic integral with respect to a Lévy process.** Now let X be a square-integrable Lévy process with semimartingale decomposition $X_t = M_t + bt$, where M is a square-integrable Lévy martingale and $b \in \mathbb{R}$. According to [8, Def. 3.7] we set

$$\text{(G-)}(\Phi \cdot X) := \text{(G-)}(\Phi \cdot M) + b\text{(G-)}(\Phi \cdot \lambda).$$

As a direct consequence of our previous results, we obtain:

2.9. **Proposition.** *For each $\Phi \in C_{\text{ad}}[0, T]$ we have*

$$\text{(G-)}(\Phi \cdot X) = {}^p\Phi \cdot X \quad \textit{in } C_{\text{ad}}[0, T].$$

In particular, (G-)$(\Phi \cdot X)$ *has a càdlàg-version.*

Summing up, we have seen that the space $C_{\text{ad}}[0, T]$ of all adapted curves from $[0, T]$ into $L^2(\Omega; H)$ is embedded into $L^2(\mathcal{P}_T)$ via $\Phi \mapsto {}^p\Phi$, see Proposition 2.5, and that the Itô-integral ${}^p\Phi \cdot X$ is a càdlàg-version of (G-)$(\Phi \cdot X)$, see Proposition 2.9. Moreover, we have seen the relation to the predictable projection in Lemma 2.6.

We close this section with an example, which seems surprising at a first view. Let X be a standard Poisson process with values in $\mathbb{R}$. In Ex. 3.9 in [8] it is derived that

$$\text{(G-)} \int_0^t X_s dX_s = \frac{1}{2}(X_t^2 - X_t).$$

Apparently, this does not coincide with the pathwise Lebesgue-Stieltjes integral

$$\int_0^t X_s dX_s = \frac{1}{2}(X_t^2 + X_t).$$

The explanation for this seemingly inconsistency is easily provided. The process X is not predictable, whence it is not Itô-integrable, and a straightforward calculation shows that

$$\text{(G-)} \int_0^t X_s dX_s = \int_0^t X_{s-} dX_s.$$

This, however, is exactly what an application of Proposition 2.9 and Lemma 2.7 yields.

3. Solutions of stochastic partial differential equations as L^2-curves

Regarding stochastic integrals as L^2-curves provides an existence and uniqueness proof for stochastic partial differential equations. Of course, this result is well-known in the literature (see, e.g., [3, 12, 1, 11, 6]), whence we only give an outline.

Consider the stochastic partial differential equation

$$\begin{cases} dr_t & = & (Ar_t + \alpha(t, r_t))dt + \sum_{i=1}^n \sigma_i(t, r_{t-})dX_t^i \\ r_0 & = & h_0, \end{cases} \tag{3.1}$$

where $A : \mathcal{D}(A) \subset H \to H$ denotes the infinitesimal generator of a C_0-semigroup $(S_t)_{t\geq 0}$ on H, and where $X^1, \dots, X^n$ are real-valued, square-integrable Lévy processes. We assume that the standard Lipschitz conditions are satisfied.

Then, there exists a unique solution $r \in C_{\text{ad}}[0, T]$ of the variation of constants equation

$$r_t := h_0 + (\text{G-}) \int_0^t S_{t-s}\alpha(s, r_s)ds + \sum_{i=1}^n (\text{G-}) \int_0^t S_{t-s}\sigma_i(s, r_s)dX_s^i,$$

see [7] for the Wiener case and [8] for the Lévy case. It is remarkable that the proof is established by means of precisely the same arguments as in the classical Picard-Lindelöf iteration scheme for ordinary differential equations, where one works on the Banach space $C([0, T]; H)$ instead of $C_{\text{ad}}[0, T]$.

Applying Proposition 2.9 for any fixed $t \in [0, T]$, we obtain the existence of a (up to a version) unique, predictable mild solution for the SPDE

$$\begin{cases} dr_t & = & (Ar_t + \alpha(t, r_t))dt + \sum_{i=1}^n \sigma_i(t, r_t)dX_t^i \\ r_0 & = & h_0, \end{cases}$$

which, in addition, is mean-square continuous.

Observe that we have no statement on path properties of the solution. If, however, the semigroup in pseudo-contractive, i.e., there exists $\omega \in \mathbb{R}$ such that

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0$$

then the stochastic convolution (Itô-)integrals have a càdlàg-version. This can be shown by using the Kotelenez inequality (see [10]) or by using the Szőkefalvi-Nagy theorem on unitary dilations (see, e.g., [14, Thm. I.8.1], or [4, Sec. 7.2]). We refer to [12, Sec. 9.4] for an overview. In this case, we conclude that there even exists a (up to indistinguishability) unique càdlàg, adapted mild solution $(r_t)_{t\geq 0}$ for (3.1), which, in addition, is mean-square continuous.

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