VIF Vienna Institute of Finance

Working Paper Series

Working Paper No. 18

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Stefan Tappe

First version: May 2009

Current version: May 2009

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STOCHASTIC INVARIANCE OF CLOSED, CONVEX SETS WITH RESPECT TO JUMP-DIFFUSIONS

STEFAN TAPPE

Abstract. We investigate stochastic invariance of closed, convex sets with respect to homogeneous jump-diffusions. Representing the closed, convex set as a countable intersection of closed half spaces, we derive invariance conditions in terms of the characteristic triplet, which concern the interior and, in particular, the boundary of the set. Homogeneous jump-diffusions encompass solutions of stochastic differential equations and affine processes, to which we apply our results.

Key Words: homogeneous jump-diffusions, stochastic invariance, closed convex sets, affine processes.

60J25, 93E03

1. Introduction

Given a closed, convex set $C \subset \mathbb{R}^d$ and a family Y of d-dimensional stochastic processes, one is often interested in the *stochastic viability* or *invariance problem* of the set C with respect to the family Y, that is, for each starting point $y_0 \in C$ the process stays in C, see Definition 2.3 below.

In the literature, the invariance problem is well studied for continuous processes, such as solutions of stochastic differential equations or stochastic control systems driven by a Brownian motion, see [11, 4, 5, 6, 2] and others.

In this article, we consider homogeneous jump-diffusions in the spirit of [10, Sec. III.2c], also allowing discontinuities. A homogeneous jump-diffusion Y with characteristic triplet (b, c, K) is a semimartingale with state dependent characteristics (B, C, ν) of the form

$$B_t = \int_0^t b(Y_s) ds, \tag{1.1}$$

$$C_t = \int_0^t c(Y_s) ds, \tag{1.2}$$

$$\nu(dt, dx) = dt K_{Y_t}(dx), \tag{1.3}$$

see Definition 2.2 below.

This is a rich class of processes, including solutions of stochastic differential equations driven by Wiener process and Poisson random measures, as well as affine processes in the sense of [8].

In order to solve the stated invariance problem, we exploit the particular structure of closed, convex sets and understand them as countable intersections of closed half spaces, which are simple geometric objects. Indeed, a classical result from convex analysis combined with Lindelöf's Lemma shows that each *closed convex set*

Date: May 22, 2009.

The author gratefully acknowledges the support from WWTF (Vienna Science and Technology Fund).

The author is grateful to Damir Filipović and Josef Teichmann for their helpful remarks and discussions.

$C \subset \mathbb{R}^d$ has a representation of the form

$$C = \bigcap_{i \in I} H_i^+, \tag{1.4}$$

where the index set I is at most *countable* and each H_i^+ is a *closed half space*, that is

$$H_i^+ = \{y \in \mathbb{R}^d : \langle \eta_i, y - z_i \rangle \geq 0\}, \quad i \in I \tag{1.5}$$

where $\eta_i \in \mathbb{R}^d \setminus \{0\}$ is a normal vector and $z_i \in \mathbb{R}^d$, see Lemma 2.1 below for the precise statement. Then, stochastic invariance of the closed, convex set C means that the process Y stays simultaneously in each of the half spaces H_i^+.

Using the representation (1.4) and introducing the edges

$$\partial C_i := \{y \in C : \langle \eta_i, y - z_i \rangle = 0\}, \quad i \in I$$

we will derive in this text the invariance conditions

$$\int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_y(dx) < \infty, \quad i \in I, \, y \in \partial C_i \tag{1.6}$$

$$\langle \eta_i, b(y) \rangle - \int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_y(dx) \geq 0, \quad i \in I, \, y \in \partial C_i \tag{1.7}$$

$$\langle \eta_i, c(y)\eta_i \rangle = 0, \quad i \in I, \, y \in \partial C_i \tag{1.8}$$

$$y + x \in C, \quad y \in C \text{ and } K_y\text{-almost all } x \in \mathbb{R}^d \tag{1.9}$$

which are formulated in terms of the characteristic triplet (b, c, K). These conditions turn out to be necessary for stochastic invariance of the closed, convex set C with respect to the family Y, but, in general, they are not sufficient. This is due to the continuous martingale part, which, in the particular case of stochastic differential equations, leads to a Stratonovich correction term of the drift.

However, in several cases, which we present in this paper, conditions (1.6), (1.7), (1.8), (1.9) are also sufficient for stochastic invariance of the closed, convex set C with respect to the family Y.

In view of (1.6), observe that condition (1.9) implies

$$\langle \eta_i, x \rangle \geq 0, \quad i \in I, \, y \in \partial C_i \text{ and } K_y\text{-almost all } x \in \mathbb{R}^d. \tag{1.10}$$

The above conditions (1.6), (1.7), (1.8) show that the behaviour of Y at the boundary of C is crucial. Condition (1.6) concerns the "small" jumps of Y and means, intuitively speaking, that, at the edge ∂C_i, the purely discontinuous martingale part must be of finite variation, unless it is parallel to the boundary

$$\partial H_i = \{y \in \mathbb{R}^d : \langle \eta_i, y - z_i \rangle = 0\}$$

of the half space H_i^+. Condition (1.6) allows us to split the purely discontinuous martingale part into two integrals, with one of them being a continuous finite variation process, which we add to the drift, and (1.7) says that this corrected drift term must be inward pointing at the edge ∂C_i. Taking into account that $\langle \eta_i, c(Y)\eta_i \rangle$ is the predictable quadratic covariation of the process $\langle \eta_i, Y \rangle$, condition (1.8) means that the diffusion part must be parallel to the boundary ∂H_i at the edge ∂C_i. Condition (1.9) is a global condition concerning also the "large" jumps of Y. It says that the closed, convex set C must capture all possible jumps of the process Y.

The remainder of this text is organized as follows. In Section 2 we provide the general setup and notations. Afterwards, we present our results on stochastic invariance for homogeneous jump-diffusions in Section 3. We apply our results to affine processes in Section 4.

2. General setup and notation

In this section, we provide the general setup and auxiliary results, we shall require in the sequel.

Let $d \in \mathbb{N}$ be a positive integer and let $C \subset \mathbb{R}^d$ be an arbitrary closed, convex set. The following result will be crucial for our investigations on stochastic invariance.

2.1. **Lemma.** *There exist an at most countable index set I and sequences $(\eta_i)_{i \in I} \subset \mathbb{R}^d \setminus \{0\}$, $(z_i)_{i \in I} \subset \mathbb{R}^d$ such that C has the representation (1.4), where the H_i^+ are given by (1.5).*

Proof. According to [12, Thm. 11.5], there exist an index set J and sequences $(\eta_j)_{j \in J} \subset \mathbb{R}^d \setminus \{0\}$, $(z_j)_{j \in J} \subset \mathbb{R}^d$ such that $C = \bigcap_{j \in J} H_j^+$ with the H_j^+ given by (1.5). We can write the complement of C as

$$\mathbb{R}^d \setminus C = \mathbb{R}^d \setminus \bigcap_{j \in J} H_j^+ = \bigcup_{j \in J} (\mathbb{R}^d \setminus H_j^+).$$

Thus, $\bigcup_{j \in J} (\mathbb{R}^d \setminus H_j^+)$ is an open covering of $\mathbb{R}^d \setminus C$. Since $\mathbb{R}^d$ is a second countable space, Lindelöf's Lemma [1, Lemma 1.1.6] yields the existence of a countable subset $I \subset J$ such that $\mathbb{R}^d \setminus C = \bigcup_{i \in I} (\mathbb{R}^d \setminus H_i^+)$, and hence we have (1.4). □

We shall now approach the definition of a *homogeneous jump-diffusion*. We denote by Sem^d the convex cone of all symmetric nonnegative definite $d \times d$-matrices. $\mathcal{M}^d$ denotes the convex cone of all (nonnegative) measures K on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $K(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1) K(dx) < \infty$. Let $b : \mathbb{R}^d \to \mathbb{R}^d$, $c : \mathbb{R}^d \to \mathrm{Sem}^d$ and $K : \mathbb{R}^d \to \mathcal{M}^d$ be given.

From now on, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

2.2. **Definition.** *(cf.* [10, Def. III.2.18]*) A process Y is called a* homogeneous jump-diffusion *with characteristic triplet (b, c, K) if it is a semimartingale with the characteristics (B, C, ν) given by (1.1), (1.2), (1.3) with respect to the truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$, $h(x) = x \mathbb{1}_{\{\|x\| \leq 1\}}$.*

According to [10, Thm. II.2.34], a homogeneous jump-diffusion Y with characteristic triplet (b, c, K) has the canonical representation

$$\begin{aligned} Y_t = Y_0 + Y_t^c &+ \int_0^t \int_{\{\|x\| \leq 1\}} x (\mu^Y(ds, dx) - K_{Y_s}(dx) ds) \\ &+ \int_0^t \int_{\{\|x\| > 1\}} x \mu^Y(ds, dx) + \int_0^t b(Y_s) ds, \quad t \geq 0 \end{aligned} \tag{2.1}$$

with respect to the truncation function h.

We proceed with the definition of *stochastic invariance*. Let $\mathcal{Y} \subset L^0(\mathcal{F}_0)$ be a subset such that $\mathcal{Y}$ contains the maps $\omega \mapsto y$, $\omega \in \Omega$ for all $y \in C$. Moreover, let $Y = (Y^{y_0})_{y_0 \in \mathcal{Y}}$ be a family of homogeneous jump-diffusions with characteristic triplet (b, c, K) and $Y_0^{y_0} = y_0$ for all $y_0 \in \mathcal{Y}$.

2.3. **Definition.** *The closed, convex set $C \subset \mathbb{R}^d$ is called* invariant *with respect to the family Y of homogeneous jump-diffusions if $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{Y_t^{y_0} \in C\}) = 1$ for all $y_0 \in \mathcal{Y}$ with $\mathbb{P}(y_0 \in C) = 1$.*

We close this section with an auxiliary result on stochastic invariance which we will require later. Let Y be a homogeneous jump-diffusion with characteristic triplet (b, c, K). For an arbitrary $\delta \geq 1$, the process Y^δ defined as

$$Y_t^\delta := Y_t - \Delta Y_t \mathbb{1}_{\{\|\Delta Y_t\| > \delta\}}, \quad t \geq 0 \tag{2.2}$$

is a special semimartingale with bounded jumps. By (2.1), we obtain the canonical decomposition

(2.3)
$$Y_t^\delta = Y_0 + Y_t^c + \int_0^t \int_{\{\|x\| \leq \delta\}} x(\mu^Y(ds,dx) - K_{Y_s}(dx)ds) + \int_0^t b^\delta(Y_s)ds, \quad t \geq 0$$

where $b^\delta : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$b^\delta(y) = b(y) + \int_{\{1 < \|x\| \leq \delta\}} x K_y(dx), \quad y \in \mathbb{R}^d.$$

Let ς^δ be the strictly positive stopping time

$$\varsigma^\delta := \inf\{t > 0 : \|\Delta Y_t\| > \delta\} = \inf\{t > 0 : Y_t - Y_t^\delta > \delta\}.$$

2.4. **Lemma.** *Let $C \subset \mathbb{R}^d$ be a closed (not necessarily convex) set. Then we have $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{Y_t \in C\}) = 1$ if and only if $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{Y^\delta_{t \wedge \varsigma^\delta} \in C\}) = 1$ for all $\delta \geq 1$.*

Proof. Assume $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{Y_t \in C\}) = 1$ and let $\delta \geq 1$ be arbitrary. Then we have $Y = Y^\delta$ on $[0, \varsigma^\delta)$ almost surely. By the closedness of C, we deduce $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{Y^\delta_{t \wedge \varsigma^\delta} \in C\}) = 1$.

Conversely, assume that $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{Y^\delta_{t \wedge \varsigma^\delta} \in C\}) = 1$ for all $\delta \geq 1$. Then we have $\mathbb{P}(\tilde{\Omega}) = 1$, where

$$\tilde{\Omega} := \bigcup_{\delta \in \mathbb{N}} \bigcap_{t \in \mathbb{R}_+} \{Y^\delta_{t \wedge \varsigma^\delta} \in C\}.$$

Let $T \in \mathbb{R}_+$ and $\omega \in \tilde{\Omega}$ be arbitrary. Since the trajectory $Y(\omega)$ is càdlàg, it only makes finitely many jumps larger than 1 on the compact interval $[0, T]$. Hence, there exists $\delta(\omega) \in \mathbb{N}$ such that $Y_t(\omega) = Y^{\delta(\omega)}_{t \wedge \varsigma^{\delta(\omega)}}(\omega) \in C$ for all $t \in [0, T]$. Since $T \in \mathbb{R}_+$ and $\omega \in \tilde{\Omega}$ were arbitrary, we deduce that $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{Y_t \in C\}) = 1$. □

3. Stochastic invariance with respect to homogeneous jump-diffusions

In this section, we derive necessary and sufficient conditions for stochastic invariance of the closed, convex set $C \subset \mathbb{R}^d$ with respect to the family Y of homogeneous jump-diffusions with characteristic triplet (b, c, K). For the set C, we take a representation of the form (1.4) with an at most countable index set I, which, according to Lemma 2.1, always exists.

For the upcoming Theorem 3.2 we impose the following continuity conditions.

3.1. **Assumption.** *We assume that the maps $b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \to \mathrm{Sem}^d$ are continuous and that $y \mapsto \int_B x K_y(dx)$ is continuous on $\mathbb{R}^d$ for all $B \in \mathcal{B}(\mathbb{R}^d)$ with $B \subset \{x \in \mathbb{R}^d : \epsilon < \|x\| \leq \delta\}$ for some $0 < \epsilon < \delta$.*

3.2. **Theorem.** *Suppose Assumption 3.1 is fulfilled. If the closed, convex set C is invariant with respect to family Y, then we have (1.6), (1.7), (1.8), (1.9).*

3.3. **Remark.** *If we have $\int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|) K_y(dx) < \infty$ for all $y \in \mathbb{R}^d$, then the family Y consists of special semimartingales and we can replace (1.6), (1.7) by the two equivalent conditions*

$$\int_{\mathbb{R}^d} \langle \eta_i, x \rangle K_y(dx) < \infty, \quad i \in I, \, y \in \partial C_i \tag{3.1}$$

$$\langle \eta_i, a(y) \rangle - \int_{\mathbb{R}^d} \langle \eta_i, x \rangle K_y(dx) \geq 0, \quad i \in I, \, y \in \partial C_i \tag{3.2}$$

where $a : \mathbb{R}^d \to \mathbb{R}^d$ *defined as*

$$a(y) := b(y) + \int_{\{\|x\|>1\}} x K_y(dx), \quad y \in \mathbb{R}^d$$

induces the finite variation part in the canonical decomposition.

Proof. Let $y_0 \in C$ be arbitrary. We set $Y := Y^{y_0}$. According to Lemma 2.4 we have

$$\mathbb{P}(Y^\delta_\tau \in C) = 1 \tag{3.3}$$

for each $\delta \geq 1$ and every finite stopping time $\tau \leq \varsigma^\delta$.

Let $\delta \geq 1$ be fixed and set $\tau_0 := \varsigma^\delta$. Let $\phi \in \mathbb{R}^d$ be constant and let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a measurable function of the form $\psi = c\mathbb{1}_B$ with $c > -1$ and $B \in \mathcal{B}(\mathbb{R}^d)$ with $B \subset \{x \in \mathbb{R}^d : \epsilon < \|x\| \leq \delta\}$ for some $\epsilon > 0$. Let Z^δ be the Doléans-Dade Exponential

$$Z^\delta_t = \mathcal{E}\bigg(\langle \phi, Y^c \rangle + \int_0^\bullet \int_{\mathbb{R}^d} \psi(x)(\mu^Y(ds,dx) - K_{Y_s}(dx)ds) \bigg)_t, \quad t \geq 0.$$

By [10, Thm. I.4.61] the process Z^δ is a solution of

$$Z^\delta_t = 1 + \sum_{j=1}^d \phi_j \int_0^t Z^\delta_s dY^{j,c}_s + \int_0^t \int_{\mathbb{R}^d} Z^\delta_{s-} \psi(x)(\mu^Y(ds,dx) - K_{Y_s}(dx)ds), \quad t \geq 0$$

and, since $\psi > -1$, the process Z is a strictly positive local martingale. There exists a strictly positive stopping time τ_1 such that $(Z^\delta)^{\tau_1}$ is a martingale. For an arbitrary $i \in I$, integration by parts yields (see [10, Thm. I.4.52])

$$\begin{aligned} \langle \eta_i, Y^\delta_t - z_i \rangle Z^\delta_t &= \int_0^t \langle \eta_i, Y^\delta_{s-} - z_i \rangle dZ^\delta_s + \int_0^t Z^\delta_{s-} d\langle \eta_i, Y^\delta - z_i \rangle_s \\ &\quad + \langle \langle \eta_i, Y^\delta - z_i \rangle^c, (Z^\delta)^c \rangle_t + \sum_{s \leq t} \Delta \langle \eta_i, Y^\delta - z_i \rangle_s \Delta Z^\delta_s, \quad t \geq 0. \end{aligned} \tag{3.4}$$

Note that, by the canonical decomposition (2.3),

$$\langle \langle \eta_i, Y^\delta - z_i \rangle^c, (Z^\delta)^c \rangle_t = \int_0^t Z^\delta_s \langle \phi, c(Y_s)\eta_i \rangle ds, \quad t \geq 0 \tag{3.5}$$

$$\sum_{s \leq t} \Delta \langle \eta_i, Y^\delta - z_i \rangle_s \Delta Z^\delta_s = \int_0^t \int_{\mathbb{R}^d} Z^\delta_{s-} \psi(x) \langle \eta_i, x \rangle \mu^Y(ds,dx), \quad t \geq 0. \tag{3.6}$$

Incorporating (2.3), (3.5) and (3.6) into (3.4), we obtain

$$\begin{aligned} \langle \eta_i, Y^\delta_t - z_i \rangle Z^\delta_t = M_t + \int_0^t Z^\delta_{s-} \bigg(&\langle \eta_i, b(Y_{s-}) \rangle + \int_{\{1<\|x\|\leq\delta\}} \langle \eta_i, x \rangle K_{Y_{s-}}(dx) \\ &+ \langle \phi, c(Y_{s-})\eta_i \rangle + \int_{\mathbb{R}^d} \psi(x) \langle \eta_i, x \rangle K_{Y_{s-}}(dx) \bigg) ds, \quad t \geq 0 \end{aligned} \tag{3.7}$$

where M is a local martingale with $M_0 = 0$. There exists a strictly positive stopping time τ_2 such that M^{τ_2} is a martingale.

By Assumption 3.1 there exist strictly positive stopping times τ_3, τ_4, τ_5 and constants $\tilde{b}, \tilde{c}(\phi), \tilde{K}(\psi) > 0$ such that

$$\bigg| \langle \eta_i, b(Y_{(t\wedge\tau_3)-}) \rangle + \int_{\{1<\|x\|\leq\delta\}} \langle \eta_i, x \rangle K_{Y_{(t\wedge\tau_3)-}}(dx) \bigg| \leq \tilde{b}, \quad t \geq 0 \tag{3.8}$$

$$|\langle \phi, c(Y_{(t\wedge\tau_4)-})\eta_i \rangle| \leq \tilde{c}(\phi), \quad t \geq 0 \tag{3.9}$$

$$\bigg| \int_{\mathbb{R}^d} \psi(x) \langle \eta_i, x \rangle K_{Y_{(t\wedge\tau_5)-}}(dx) \bigg| \leq \tilde{K}(\psi), \quad t \geq 0. \tag{3.10}$$

Let $B := \{x \in \mathbb{R}^d : y_0 + x \notin C\}$. In order to prove (1.9), it suffices, since $y_0 \in C$, to show that $K_{y_0}(B \cap \{x \in \mathbb{R}^d : \epsilon < \|x\| \leq \delta\}) = 0$ for all $0 < \epsilon < \delta$. Suppose, on the contrary, there exist $0 < \epsilon < \delta$ such that $K_{y_0}(B \cap \{x \in \mathbb{R}^d : \epsilon < \|x\| \leq \delta\}) > 0$. By countability of I there exists $i \in I$ such that $K_{y_0}(B_i \cap \{x \in \mathbb{R}^d : \epsilon < \|x\| \leq \delta\}) > 0$, where $B_i := \{x \in \mathbb{R}^d : y_0 + x \notin H_i^+\}$. We obtain

$$\int_{B_i \cap \{\epsilon < \|x\| \leq \delta\}} \langle \eta_i, x \rangle K_{y_0}(dx) \leq \int_{B_i \cap \{\epsilon < \|x\| \leq \delta\}} \langle \eta_i, y_0 + x \rangle K_{y_0}(dx) < 0.$$

By Assumption 3.1, there exist $\kappa > 0$ and a strictly positive stopping time $\tau_6 \leq 1$ such that

$$\int_{B_i \cap \{\epsilon < \|x\| \leq \delta\}} \langle \eta_i, x \rangle K_{Y_{(t \wedge \tau_6)-}}(dx) \leq -\kappa, \quad t \geq 0.$$

Let $\phi := 0$, $\psi := \frac{\tilde{b}+1}{\kappa} \mathbb{1}_{B_i \cap \{\epsilon < \|x\| \leq \delta\}}$ and $\tau := \bigwedge_{i=0}^6 \tau_i$. Taking expectation in (3.7) we obtain $\mathbb{E}[\langle \eta_i, Y_\tau^\delta - z_i \rangle Z_\tau^\delta] < 0$, implying $\mathbb{P}(\langle \eta_i, Y_\tau^\delta - z_i \rangle < 0) > 0$, which contradicts (3.3). This yields (1.9).

From now on, we assume that $y_0 \in \partial C_i$ for an arbitrary $i \in I$ and $\delta := 1$. Then equation (3.7) simplifies to

$$\begin{aligned} \langle \eta_i, Y_t^1 - z_i \rangle Z_t^1 = M_t + \int_0^t Z_{s-}^1 \bigg(&\langle \eta_i, b(Y_{s-}) \rangle + \langle \phi, c(Y_{s-}) \eta_i \rangle \\ &+ \int_{\mathbb{R}^d} \psi(x) \langle \eta_i, x \rangle K_{Y_{s-}}(dx) \bigg) ds, \quad t \geq 0 \end{aligned} \tag{3.11}$$

and relation (3.8) becomes

$$|\langle \eta_i, b(Y_{(t \wedge \tau_3)-}) \rangle| \leq \tilde{b}, \quad t \geq 0.$$

Suppose that $\langle \eta_i, c(y_0) \eta_i \rangle > 0$. By the continuity of c (see Assumption 3.1) there exist $\kappa > 0$ and a strictly positive stopping time $\tau_6 \leq 1$ such that

$$\langle \eta_i, c(Y_{(t \wedge \tau_6)-}^1) \eta_i \rangle \geq \kappa, \quad t \geq 0.$$

Let $\phi := -\frac{\tilde{b}+1}{\kappa} \eta_i$, $\psi := 0$ and $\tau := \bigwedge_{i=0}^6 \tau_i$. Taking expectation in (3.11) yields $\mathbb{E}[\langle \eta_i, Y_\tau^1 - z_i \rangle Z_\tau^1] < 0$, implying $\mathbb{P}(\langle \eta_i, Y_\tau^1 - z_i \rangle < 0) > 0$, which contradicts (3.3). This proves (1.8).

Now suppose $\int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_{y_0}(dx) = \infty$. Using Assumption 3.1 and noting relation (1.10), there exist $\epsilon \in (0,1)$ and a strictly positive stopping time $\tau_6 \leq 1$ such that

$$-\frac{1}{2} \int_{\{\epsilon < \|x\| \leq 1\}} \langle \eta_i, x \rangle K_{Y_{(t \wedge \tau_6)-}}(dx) \leq -(\tilde{b} + 1), \quad t \geq 0.$$

Let $\phi := 0$, $\psi := -\frac{1}{2} \mathbb{1}_{\{\epsilon < \|x\| \leq 1\}}$ and $\tau := \bigwedge_{i=0}^6 \tau_i$. Taking expectation in (3.11) yields $\mathbb{E}[\langle \eta_i, Y_\tau^1 - z_i \rangle Z_\tau^1] < 0$, implying $\mathbb{P}(\langle \eta_i, Y_\tau^1 - z_i \rangle < 0) > 0$, which contradicts (3.3). This yields (1.6).

Next, we show for all $n \in \mathbb{N}$ the relation

$$\langle \eta_i, b(y_0) \rangle + \int_{\mathbb{R}^d} \psi_n(x) \langle \eta_i, x \rangle K_{y_0}(dx) \geq 0, \tag{3.12}$$

where $\psi_n := -(1 - \frac{1}{n}) \mathbb{1}_{\{\frac{1}{n} < \|x\| \leq 1\}}$. Suppose, on the contrary, that (3.12) is not satisfied for some $n \in \mathbb{N}$. Using Assumption 3.1, there exist $\kappa > 0$ and a strictly positive stopping time $\tau_6 \leq 1$ such that

$$\langle \eta_i, b(Y_{(t \wedge \tau_6)-}) \rangle + \int_{\mathbb{R}^d} \psi_n(x) \langle \eta_i, x \rangle K_{Y_{(t \wedge \tau_6)-}}(dx) \leq -\kappa, \quad t \geq 0.$$

Let $\phi := 0$ and $\tau := \bigwedge_{i=0}^{6} \tau_i$. Taking expectation in (3.11) we obtain $\mathbb{E}[\langle \eta_i, Y_\tau^1 - z_i \rangle Z_\tau^1] < 0$, implying $\mathbb{P}(\langle \eta_i, Y_\tau^1 - z_i \rangle < 0) > 0$, which contradicts (3.3). This yields (3.12). By (3.12), (1.6) and Lebesgue's theorem, we conclude (1.7). □

We shall now provide sufficient conditions for stochastic invariance. For this purpose, we introduce the reflected half spaces

$$H_i^- := \{y \in \mathbb{R}^d : \langle \eta_i, y - z_i \rangle \leq 0\}, \quad i \in I.$$

3.4. **Theorem.** *Suppose we have*

$$\int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_y(dx) < \infty, \quad i \in I,\, y \in H_i^- \tag{3.13}$$

$$\langle \eta_i, b(y) \rangle - \int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_y(dx) \geq 0, \quad i \in I,\, y \in H_i^- \tag{3.14}$$

$$\langle \eta_i, c(y)\eta_i \rangle = 0, \quad i \in I,\, y \in H_i^- \tag{3.15}$$

$$\langle \eta_i, x \rangle \geq 0, \quad i \in I,\, y \in H_i^- \text{ and } K_y\text{-almost all } x \in \mathbb{R}^d \tag{3.16}$$

$$\langle \eta_i, y + x - z_i \rangle \geq 0, \quad i \in I,\, y \in H_i^+ \text{ and } K_y\text{-almost all } x \in \mathbb{R}^d. \tag{3.17}$$

Then, the closed, convex set C is invariant with respect to the family Y.

Proof. Suppose the closed, convex set C is not invariant with respect to the family Y. By Lemma 2.4, there exist $y_0 \in \mathcal{Y}$ with $\mathbb{P}(y_0 \in C) = 1$, an index $i \in I$ and $\delta \geq 1$ such that $\mathbb{P}(\tau < \infty) > 0$, where

$$\tau := \inf\{t > 0 : \langle \eta_i, Y_{t \wedge \varsigma^\delta}^\delta - z_i \rangle < 0\}$$

and Y^δ denotes the special semimartingale defined in (2.2) with $Y := Y^{y_0}$. Using Itô's formula [10, Thm. I.4.57] shows that the process $\langle \eta_i, Y^c \rangle$ has the predictable quadratic covariation

$$\langle \langle \eta_i, Y^c \rangle, \langle \eta_i, Y^c \rangle \rangle = \langle \eta_i, c(Y)\eta_i \rangle, \quad i \in I. \tag{3.18}$$

Noting that $\{\tau < \infty\} = \bigcup_{n \in \mathbb{N}} \{\tau_n < \infty\}$, where

$$\tau_n := \inf\Big\{t > 0 : \langle \eta_i, Y_{t \wedge \varsigma^\delta}^\delta - z_i \rangle \leq -\frac{1}{n}\Big\}, \quad n \in \mathbb{N}$$

there exists $n \in \mathbb{N}$ such that $\mathbb{P}(\tau_n < \infty) > 0$. Because of (3.17) we have

$$\mathbb{P}\bigg(\bigcup_{m \in \mathbb{N}} \bigcap_{t \in [\tau_n - \frac{1}{m}, \tau_n] \cap \mathbb{Q}_+} \{\langle \eta_i, Y_{t \wedge \varsigma^\delta}^\delta - z_i \rangle < 0\} \bigg) > 0,$$

and hence there exists $m \in \mathbb{N}$ such that

$$\mathbb{P}\bigg(\bigcap_{t \in [\tau_n - \frac{1}{m}, \tau_n] \cap \mathbb{R}_+} \{\langle \eta_i, Y_{t \wedge \varsigma^\delta}^\delta - z_i \rangle < 0\} \bigg) > 0.$$

Choosing $\epsilon := \frac{1}{n}$ and $\kappa := \frac{1}{m}$ we have $\mathbb{P}(\tau_{\epsilon,\kappa} < \infty) > 0$, where

$$\begin{aligned} \tau_{\epsilon,\kappa} := \inf\{t > 0 : {} & \langle \eta_i, Y_{t \wedge \varsigma^\delta}^\delta - z_i \rangle \leq -\epsilon \text{ and} \\ & \langle \eta_i, Y_{s \wedge \varsigma^\delta}^\delta - z_i \rangle \in (-\epsilon, 0) \text{ for all } s \in [t - \kappa, t] \cap \mathbb{R}_+\}. \end{aligned}$$

On $\{\tau_{\epsilon,\kappa} < \infty\}$ we obtain by the canonical decomposition (2.3) and relations (3.13), (3.14), (3.15), (3.16), (3.18)

$$\begin{aligned} 0 &> \langle \eta_i, Y^\delta_{\tau_{\epsilon,\kappa} \wedge \varsigma^\delta} - z_i \rangle - \langle \eta_i, Y^\delta_{(\tau_{\epsilon,\kappa} - \kappa)^+ \wedge \varsigma^\delta} - z_i \rangle = Y^c_{\tau_{\epsilon,\kappa} \wedge \varsigma^\delta} - Y^c_{(\tau_{\epsilon,\kappa} - \kappa)^+ \wedge \varsigma^\delta} \\ &+ \int_{(\tau_{\epsilon,\kappa} - \kappa)^+ \wedge \varsigma^\delta}^{\tau_{\epsilon,\kappa} \wedge \varsigma^\delta} \int_{\{\|x\| \leq \delta\}} \langle \eta_i, x \rangle \mu^Y(ds, dx) \\ &+ \int_{(\tau_{\epsilon,\kappa} - \kappa)^+ \wedge \varsigma^\delta}^{\tau_{\epsilon,\kappa} \wedge \varsigma^\delta} \bigg(\langle \eta_i, b(Y_s) \rangle - \int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_{Y_s}(dx) \bigg) ds \geq 0, \end{aligned}$$

a contradiction. □

Let $\Pi : \mathbb{R}^d \to C$ be the metric projection on the closed, convex set C. For every $y_0 \in \mathbb{R}^d$, the metric projection $\Pi(y_0)$ is given by the unique element $y \in C$ such that

$$\|y - y_0\| = \inf_{z \in C} \|z - y_0\|,$$

see, e.g., [13, Satz V.3.2].

Now, we assume that $Y = (Y^{y_0})_{y_0 \in \mathcal{Y}}$ is a family of homogeneous jump-diffusions with characteristic triplet $(b \circ \Pi, c \circ \Pi, K \circ \Pi)$.

3.5. **Corollary.** *Suppose we have*

$$\Pi(y) \in \partial C_i, \quad i \in I,\, y \in H_i^- \tag{3.19}$$

$$\langle \eta_i, y \rangle = \langle \eta_i, \Pi(y) \rangle, \quad i \in I,\, y \in H_i^+ \tag{3.20}$$

and conditions (1.6), (1.7), (1.8), (1.9) are fulfilled. Then, the closed, convex set C is invariant with respect to the family Y. In particular, Y is a family of homogeneous jump-diffusions with characteristic triplet (b, c, K).

Proof. Using (1.6), (1.7), (1.8), (1.9) and (3.19), (3.20) we obtain

$$\begin{aligned} &\int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_{\Pi(y)}(dx) < \infty, \quad i \in I,\, y \in H_i^- \\ &\langle \eta_i, b(\Pi(y)) \rangle - \int_{\{\|x\| \leq 1\}} \langle \eta_i, x \rangle K_{\Pi(y)}(dx) \geq 0, \quad i \in I,\, y \in H_i^- \\ &\langle \eta_i, c(\Pi(y)) \eta_i \rangle = 0, \quad i \in I,\, y \in H_i^- \\ &\langle \eta_i, x \rangle = \langle \eta_i, \Pi(y) + x - z_i \rangle \geq 0, \quad i \in I,\, y \in H_i^- \text{ and } K_{\Pi(y)}\text{-almost all } x \in \mathbb{R}^d \\ &\langle \eta_i, y + x - z_i \rangle \geq 0, \quad i \in I,\, y \in H_i^+ \text{ and } K_{\Pi(y)}\text{-almost all } x \in \mathbb{R}^d. \end{aligned}$$

Hence, Theorem 3.4 applies. □

3.6. **Remark.** *Note that conditions (3.19), (3.20) are satisfied if the closed, convex set C is a half space of the form (1.5), or a convex cone of the type $C = \mathbb{R}^m_+ \times \mathbb{R}^n$, see Section 4.*

Summing up the achievements of this section, conditions (1.6), (1.7), (1.8), (1.9) are necessary and, subject to enough regularity, also sufficient for stochastic invariance of the closed, convex set C with respect to the family Y of homogeneous jump-diffusions.

Particular examples of homogeneous jump-diffusions are solutions of stochastic differential equations

(3.21)
$$dY_t = \alpha(Y_t)dt + \sum_j \sigma^j(Y_t)d\beta_t^j + \int_{\{\|\gamma(Y_{t-},\cdot)\|\le 1\}} \gamma(Y_{t-},x)(\mu(dt,dx) - F(dx)dt) + \int_{\{\|\gamma(Y_{t-},\cdot)\|>1\}} \gamma(Y_{t-},x)\mu(dt,dx)$$

driven by a (possibly infinite dimensional) Wiener process W with covariance operator Q having the expansion

$$W = \sum_j \sqrt{\lambda_j}\beta^j e_j,$$

(see [3, Prop. 4.1]), and a homogeneous Poisson random measure μ on $\mathbb{R}_+ \times E$, see [10, Def. II.1.20], with compensator $dt \otimes F(dx)$, on the mark space $(E, \mathcal{E})$, which is allowed to be a Blackwell space (see [7, 9]). Under appropriate regularity assumptions, the solutions of (3.21) form a family of homogeneous jump-diffusions with characteristic triplet given by

$$b(y) = \alpha(y), \quad y \in \mathbb{R}^d$$
$$c(y) = \sigma(y)Q^{\frac{1}{2}}(\sigma(y)Q^{\frac{1}{2}})^*, \quad y \in \mathbb{R}^d$$
$$K_y(B) = \int_{\mathbb{R}^d} \mathbb{1}_{B\setminus\{0\}}(\gamma(y,x))F(dx), \quad y \in \mathbb{R}^d \text{ and } B \in \mathcal{B}(\mathbb{R}^d).$$

As mentioned above, the derived invariance conditions are, in general, only necessary, because, in general, the drift has to compensated by a Stratonovich correction term, see, e.g., [6]. However, if the conditions of Theorem 3.4 or Corollary 3.5 are fulfilled, then the Stratonovich term vanishes.

Homogeneous jump-diffusions also encompass affine processes, which are tailor-made for an application of our previous results. We study them in the upcoming section.

4. Stochastic invariance with respect to affine processes

In this section, we apply our preceding results to affine processes.

As before, let $d \in \mathbb{N}$ be a positive integer. Let $m, n \in \mathbb{N}_0$ be such that $m + n = d$. We set $C = \mathbb{R}_+^m \times \mathbb{R}^n$, which is a closed, convex set of the form (1.4) with $I = \{1, \dots, m\}$ and $\eta_i = e_i$, $z_i = 0$ for $i = 1, \dots, m$. Here the e_i, $i = 1, \dots, d$ denote the unit vectors in $\mathbb{R}^d$.

Note that C is also a convex cone. Its edges are given by

$$\partial C_i = \{y \in C : y_i = 0\}, \quad i = 1, \dots, m.$$

Let $Y = (Y^{y_0})_{y_0 \in \mathcal{Y}}$ be a family of homogeneous jump-diffusions with characteristic triplet (b, c, K). Using the particular form of the convex cone C, the following auxiliary result is obvious.

4.1. **Lemma.** *Conditions (1.6), (1.7), (1.8), (1.9) are satisfied if and only if*

$$\int_{\{\|x\|\le 1\}} x_i K_y(dx) < \infty, \quad i = 1, \dots, m, \; y \in \partial C_i \tag{4.1}$$

$$b_i(y) - \int_{\{\|x\|\le 1\}} x_i K_y(dx) \ge 0, \quad i = 1, \dots, m, \; y \in \partial C_i \tag{4.2}$$

$$c_{ii}(y) = 0, \quad i = 1, \dots, m, \; y \in \partial C_i \tag{4.3}$$

$$y + x \in C, \quad y \in C \text{ and } K_y\text{-almost all } x \in \mathbb{R}^d. \tag{4.4}$$

We assume that the characteristic triplet (b, c, K) is of the affine form

$$b_i(y) = b_i^0 + \int_{\{\|x\| \leq 1\}} x_i K^0(dx) + \sum_{j=1}^m b_{ij}^1 y_j^+ + \sum_{\substack{j=1 \\ j \neq i}}^m y_j^+ \int_{\{\|x\| \leq 1\}} x_i K^j(dx) + \sum_{j=m+1}^d b_{ij}^1 y_j, \quad y \in \mathbb{R}^d \text{ and } i = 1, \ldots, m \tag{4.5}$$

$$b_i(y) = b_i^0 + \sum_{j=1}^m b_{ij}^1 y_j^+ + \sum_{j=m+1}^d b_{ij}^1 y_j, \quad y \in \mathbb{R}^d \text{ and } i = m+1, \ldots, d \tag{4.6}$$

$$c(y) = c^0 + \sum_{j=1}^m y_j^+ c^j, \quad y \in \mathbb{R}^d \tag{4.7}$$

$$K_y(B) = K^0(B) + \sum_{j=1}^m y_j^+ K^j(B), \quad y \in \mathbb{R}^d \text{ and } B \in \mathcal{B}(\mathbb{R}^d) \tag{4.8}$$

where $b^0 \in \mathbb{R}^d$, $b^1 \in \mathbb{R}^{d \times d}$ and $c^i \in \mathrm{Sem}^d$, $K^i \in \mathcal{M}^d$ for $i = 0, \ldots, m$. We agree to set $b_i(y) = \infty$ if one of the integrals in (4.5) diverges. Note that c maps into Sem^d and K maps into $\mathcal{M}^d$, because Sem^d, $\mathcal{M}^d$ are convex cones.

Because of the affine structure of the characteristic triplet (b, c, K), we are, in the sequel, concerned with *affine* processes.

In the spirit of [8, Def. 2.6] we make the following definition.

4.2. **Definition.** *We call the characteristic triplet* (b, c, K) admissible *if the following conditions are satisfied:*

- *We have* $b^0 \in C$, *i.e.*

$$b_i^0 \geq 0, \quad i = 1, \ldots, m. \tag{4.9}$$

- *For each* $i = 1, \ldots, m$ *we have*

$$b_{ij}^1 \geq 0, \quad j \in \{1, \ldots, m\} \setminus \{i\}, \tag{4.10}$$

$$b_{ij}^1 = 0, \quad j = m+1, \ldots, d. \tag{4.11}$$

- *We have*

$$c_{ii}^0 = 0, \quad i = 1, \ldots, m. \tag{4.12}$$

- *For each* $i = 1, \ldots, m$ *we have*

$$c_{ii}^j = 0, \quad j \in \{1, \ldots, m\} \setminus \{i\}. \tag{4.13}$$

- *We have* $\mathrm{supp}(K^0) \subset C$ *and*

$$\int_{\{\|x\| \leq 1\}} x_i K^0(dx) < \infty, \quad i = 1, \ldots, m. \tag{4.14}$$

- *For each* $i = 1, \ldots, m$ *we have* $\mathrm{supp}(K^i) \subset C$ *and*

$$\int_{\{\|x\| \leq 1\}} x_i K^j(dx) < \infty, \quad j \in \{1, \ldots, m\} \setminus \{i\}. \tag{4.15}$$

4.3. **Proposition.** *The characteristic triplet* (b, c, K) *is admissible if and only if conditions (4.1)–(4.4) are satisfied.*

Proof. "$\Rightarrow$": Note that the integrals in (4.5) converge because of (4.14) and (4.15).

In the sequel, let $i \in \{1, \dots, m\}$ and $y \in \partial C_i$ be arbitrary. By (4.12) and (4.13) we obtain

$$c_{ii}(y) = c_{ii}^0(y) + \sum_{\substack{j=1 \\ j \neq i}}^m y_j c_{ii}^j = 0,$$

showing (4.3). Using (4.14) and (4.15) we get

$$\int_{\{\|x\| \leq 1\}} x_i K_y(dx) = \int_{\{\|x\| \leq 1\}} x_i K^0(dx) + \sum_{\substack{j=1 \\ j \neq i}}^m y_j \int_{\{\|x\| \leq 1\}} x_i K^j(dx) < \infty,$$

proving (4.1). Relations (4.9), (4.10) and (4.11) yield

$$b_i(y) - \int_{\{\|x\| \leq 1\}} x_i K_y(dx) = b_i^0 + \sum_{\substack{j=1 \\ j \neq i}}^m b_{ij}^1 y_j + \sum_{j=m+1}^d b_{ij}^1 y_j \geq 0,$$

which proves (4.2).

Note that $\operatorname{supp}(K_y) \subset C$ for all $y \in C$, because $\operatorname{supp}(K^i) \subset C$ for all $i = 0, \dots, m$. Consequently, (4.4) is satisfied, because C is a convex cone.

"$\Leftarrow$": By (4.3) we have

$$c_{ii}^0 = c_{ii}(0) = 0, \quad i = 1, \dots, m$$

showing (4.12). For each $i = 1, \dots, m$ relations (4.12) and (4.3) give us

$$c_{ii}^j = c_{ii}^0 + c_{ii}^j = c_{ii}(e_j) = 0, \quad j \in \{1, \dots, m\} \setminus \{i\}$$

proving (4.13).

Putting $y = 0$ in (4.4) we obtain $\operatorname{supp}(K^0) = \operatorname{supp}(K_0) \subset C$. Suppose, for some $i \in \{1, \dots, m\}$ there exists $x \in \operatorname{supp}(K^i)$ with $x \notin C$. Then we have $x \in \operatorname{supp}(K_{\lambda e_i})$ for all $\lambda > 0$. Using (4.4) we obtain $\lambda e_i + x \in C$ for all $\lambda > 0$. By the closedness of C we arrive at the contradiction $x \in C$. Hence $\operatorname{supp}(K^i) \subset C$ for all $i = 1, \dots, m$.

By (4.1) we have

$$\int_{\{\|x\| \leq 1\}} x_i K^0(dx) = \int_{\{\|x\| \leq 1\}} x_i K_0(dx) < \infty, \quad i = 1, \dots, m$$

showing (4.14). For each $i = 1, \dots, m$ relations (4.14) and (4.1) yield

$$\int_{\{\|x\| \leq 1\}} x_i K^j(dx) = \int_{\{\|x\| \leq 1\}} x_i K_{e_j}(dx) - \int_{\{\|x\| \leq 1\}} x_i K^0(dx) < \infty$$

for all $j \in \{1, \dots, m\} \setminus \{i\}$, which proves (4.15).

In the sequel, let $i \in \{1, \dots, m\}$ be arbitrary. According to (4.5) and (4.2) we have

$$b_i^0 = b_i(0) - \int_{\{\|x\| \leq 1\}} x_i K^0(dx) = b_i(0) - \int_{\{\|x\| \leq 1\}} x_i K_0(dx) \geq 0,$$

showing (4.9). For each $j \in \{1, \dots, m\} \setminus \{i\}$ we have, by using (4.5) and (4.2),

$$\begin{aligned} b_i^0 + \lambda b_{ij}^1 &= b_i(\lambda e_j) - \int_{\{\|x\| \leq 1\}} x_i K^0(dx) - \lambda \int_{\{\|x\| \leq 1\}} x_i K^j(dx) \\ &= b_i(\lambda e_j) - \int_{\{\|x\| \leq 1\}} x_i K_{\lambda e_j}(dx) \geq 0 \quad \text{for all } \lambda \geq 0. \end{aligned}$$

Because of (4.9), we obtain (4.10). For each $j = m+1, \dots, d$ we have, by using (4.5) and (4.2) again,

$$\begin{aligned} b_i^0 + \lambda b_{ij}^1 &= b_i(\lambda e_j) - \int_{\{\|x\| \leq 1\}} x_i K^0(dx) \\ &= b_i(\lambda e_j) - \int_{\{\|x\| \leq 1\}} x_i K_{\lambda e_j}(dx) \geq 0 \quad \text{for all } \lambda \in \mathbb{R}. \end{aligned}$$

This relation proves (4.11). □

4.4. **Theorem.** *The closed, convex set C is invariant with respect to the family Y if and only if the characteristic triplet (b, c, K) is admissible.*

Proof. Since the characteristic triplet (b, c, K) is of the affine form given by (4.5), (4.6), (4.7) and (4.8), Assumption 3.1 is fulfilled. The metric projection $\Pi : \mathbb{R}^d \to C$ is given by

$$\Pi(y) = \sum_{j=1}^{m} y_j^+ e_j + \sum_{j=m+1}^{d} y_j e_j, \quad y \in \mathbb{R}^d.$$

Hence, conditions (3.19), (3.20) are fulfilled and, since the characteristic triplet (b, c, K) is of the affine form given by (4.5), (4.6), (4.7) and (4.8), we have $b \circ \Pi = b$, $c \circ \Pi = c$ and $K \circ \Pi = K$.

Combining Theorem 3.2, Corollary 3.5, Lemma 4.1 and Proposition 4.3, we arrive at the stated result. □

4.5. **Remark.** *By Definition 2.2, the family Y consists of semimartingales with characteristics (B, C, ν) given by (1.1), (1.2), (1.3). If the characteristic triplet (b, c, K) is admissible,* [8, Thm. 2.12] *shows that each Y^{y_0} is a regular, conservative, affine process with state space C.*

References

[1] Abraham, R., Marsden, J. E., Ratiu, T. (1988): *Manifolds, tensor analysis, and applications.* Berlin: Springer.

[2] Buckdahn, R., Quincampoix, M., Rainer, C., Teichmann, J. (2007): A simple proof for the equivalence between invariance for stochastic and deterministic systems. to appear in *Bulletin des Sciences Mathematiques.*

[3] Da Prato, G., Zabczyk, J. (1992): *Stochastic equations in infinite dimensions.* New York: Cambridge University Press.

[4] Da Prato, G., Frankowska, H. (2001): Stochastic viability for compact sets in terms of the distance function. *Dynamic Systems Appl.* **10**, 177–184.

[5] Da Prato, G., Frankowska, H. (2004): Invariance of stochastic control systems with deterministic arguments. *Journal of Differential Equations* **200**, 18–52.

[6] Da Prato, G., Frankowska, H. (2007): Stochastic viability of convex sets. *Journal of Mathematical Analysis and Applications* **333**, 151–163

[7] Dellacherie, C., Meyer, P. A. (1982): *Probabilités et potentiel.* Hermann: Paris.

[8] Duffie, D., Filipović, D., Schachermayer, W. (2003): Affine processes and applications in finance. *Annals of Applied Probability* **13**, 984–1053.

[9] Getoor, R. K. (1979): On the construction of kernels. *Séminaire de Probabilités* IX, Lecture Notes in Mathematics 465, 443–463.

[10] Jacod, J., Shiryaev, A. N. (2003): *Limit theorems for stochastic processes.* Springer, Berlin.

[11] Milian, A. (1995): Stochastic viability and a comparison theorem. *Coll. Math.* **LXVIII**, 297–316.

[12] Rockafellar, R. T. (1997): *Convex Analysis.* Princeton University Press.

[13] Werner, D. (2002): *Funktionalanalysis.* Fourth Edition, Berlin: Springer.

Vienna Institute of Finance, University of Vienna, and Vienna University of Economics and Business Administration, Heiligenstädter Strasse 46-48, A-1190 Wien, Austria

E-mail address: stefan.tappe@vif.ac.at