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OPTION PRICING IN BILATERAL GAMMA STOCK MODELS

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Abstract. In the framework of bilateral Gamma stock models we seek for adequate option pricing measures, which have an economic interpretation and allow numerical calculations of option prices. Our investigations encompass Esscher transforms, minimal entropy martingale measures and bilateral Esscher transforms. We illustrate our theory by a numerical example.

Key Words: Bilateral Gamma stock model, minimal entropy martingale measure, bilateral Esscher transform, option pricing.

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1. Introduction

An issue in continuous time finance is to find realistic and analytically tractable models for price evolutions of risky financial assets. In this text, we consider exponential Lévy models

$$\begin{cases} S_t &= S_0 e^{X_t} \\ B_t &= e^{rt} \end{cases} \tag{1.1}$$

consisting of two financial assets (S, B), one dividend paying stock S with dividend rate $q \geq 0$, where X denotes a Lévy process and one risk free asset B, the bank account with fixed interest rate $r \geq 0$. Note that the classical Black-Scholes model is a special case by choosing $X_t = \sigma W_t + (\mu - \frac{\sigma^2}{2})t$, where W is a Wiener process, $\mu \in \mathbb{R}$ denotes the drift and $\sigma > 0$ the volatility.

Although the Black-Scholes model is a standard model in financial mathematics, it is well-known that it only provides a poor fit to observed market data, because typically the empirical densities possess heavier tails and much higher located modes than normal distributions.

Several authors therefore suggested more sophisticated Lévy processes with a jump component. We mention the Variance Gamma processes [21, 22], hyperbolic processes [6], normal inverse Gaussian processes [1], generalized hyperbolic processes [7, 8, 23], CGMY processes [3] and Meixner processes [26]. A survey about Lévy processes used for applications to finance can for instance be found in [5, Chap. 4] or [25, Chap. 5.3].

Recently, the class of bilateral Gamma processes, which we shall henceforth deal with in this text, was proposed in [18]. We also mention the related article [19], where the shapes of their densities are investigated.

Now, let (S, B) be an exponential Lévy model of the type (1.1). In practice, we often have to deal with adequate pricing of European options $\Phi(S_T)$, where $T > 0$ denotes the time of maturity and $\Phi : \mathbb{R} \to \mathbb{R}$ the payoff function. For example, the payoff profile of a European call option is $\Phi(x) = (x - K)^+$.

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The option price is given by $e^{-rT}\mathbb{E}_{\mathbb{Q}}[\Phi(S_T)]$, where $\mathbb{Q} \sim \mathbb{P}$ is a *local martingale measure*, i.e., a probability measure, which is equivalent to the objective probability measure $\mathbb{P}$, such that the discounted stock price process

$$\tilde{S}_t := e^{-(r-q)t} S_t = S_0 e^{X_t - (r-q)t}, \quad t \geq 0 \tag{1.2}$$

is a local $\mathbb{Q}$-martingale. Typically, exponential Lévy models (1.1) are free of arbitrage, but, unlike the classical Black-Scholes model, not complete, that is, there exist several martingale measures $\mathbb{Q} \sim \mathbb{P}$. In particular, if not enough market data for calibration are available, we are therefore faced with problem, which martingale measure we should choose.

This text is devoted to the existence of suitable pricing measures $\mathbb{Q} \sim \mathbb{P}$ for bilateral Gamma stock models. We will in particular focus on the following two criteria:

- Under $\mathbb{Q}$, the process X should again be a bilateral Gamma process, because this allows, by virtue of the simple characteristic function (3.1) below, numerical calculations of option prices by using the method of Fourier transformation.
- The measure $\mathbb{Q}$ should have a reasonable economic interpretation.

We proceed as follows: In Section 2 we provide the required results from stochastic calculus. In Section 3 we review bilateral Gamma processes and introduce bilateral Gamma stock models. Sections 4, 5, 6 are devoted to several approaches on choosing suitable martingale measures for option pricing in bilateral Gamma stock models. We conclude with a numerical illustration in Section 7.

2. Prerequisites from stochastic calculus

In this section, we collect the results from stochastic calculus, which we will require in the sequel. Throughout this text, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

2.1. **Lemma.** [13, Thm. I.4.61] *Let X be a real-valued semimartingale. There exists a unique (up to indistinguishability) solution Z for the equation*

$$Z_t = 1 + \int_0^t Z_{s-} dX_s, \quad t \geq 0. \tag{2.1}$$

2.2. **Definition.** *Let X be a real-valued semimartingale. We call the unique solution Z for (2.1) the* stochastic exponential *or* Doléans-Dade exponential of X *and write* $\mathcal{E}(X) := Z$.

2.3. **Lemma.** [16, Lemma 2.2] *Let Z be a semimartingale such that Z, Z_- are $\mathbb{R} \setminus \{0\}$-valued. There there exists a unique (up to indistinguishability) semimartingale X such that $X_0 = 0$ and $Z = Z_0\mathcal{E}(X)$. It is given by*

$$X_t = \int_0^t \frac{1}{Z_{s-}} dZ_s, \quad t \geq 0. \tag{2.2}$$

2.4. **Definition.** *Let Z be a semimartingale such that Z, Z_- are $\mathbb{R} \setminus \{0\}$-valued. We call the unique process X from Lemma 2.3 the* stochastic logarithm of X *and write* $\mathcal{L}(Z) := X$.

2.5. **Lemma.** *Let X be a real-valued Lévy process. The following statements are equivalent:*

1. *X is a local martingale;*
2. *X is a martingale;*
3. $\mathbb{E}[X_1] = 0$.

Proof. For (1) $\Leftrightarrow$ (2) see [27]. Noting that $\mathbb{E}[X_t] = t\mathbb{E}[X_1]$ for all $t \geq 0$, the equivalence (2) $\Leftrightarrow$ (3) follows from [5, Prop. 3.17]. □

2.6. **Lemma.** *Let X be a real-valued Lévy process. The following statements are equivalent:*

(1) e^X *is a local martingale;*
(2) e^X *is a martingale;*
(3) $\mathbb{E}[e^{X_1}] = 1$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from [15, Lemma 4.4.3]. Noting that $\mathbb{E}[e^{X_t}] = (\mathbb{E}[e^{X_1}])^t$ for all $t \geq 0$, the equivalence (2) $\Leftrightarrow$ (3) follows from [5, Prop. 3.17]. □

3. Stock price models driven by bilateral Gamma processes

In this section, we shall introduce bilateral Gamma stock models. For this purpose, we first review the family of bilateral Gamma processes. For details and more information, we refer to [18].

A *bilateral Gamma distribution* with parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^- > 0$ is defined as the distribution of $Y - Z$, where Y and Z are independent, $Y \sim \Gamma(\alpha^+, \lambda^+)$ and $Z \sim \Gamma(\alpha^-, \lambda^-)$.

The characteristic function of a bilateral Gamma distribution is given by

$$\phi(z) = \left(\frac{\lambda^+}{\lambda^+ - iz} \right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + iz} \right)^{\alpha^-}, \quad z \in \mathbb{R} \tag{3.1}$$

where the powers stem from the main branch of the complex logarithm.

Thus, any bilateral Gamma distribution is infinitely divisible, which allows us to define its associated Lévy process $(X_t)_{t \geq 0}$, which we call a *bilateral Gamma process.* We write $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ if X_1 has a bilateral Gamma distribution with parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^- > 0$. All increments of X have a bilateral Gamma distribution, more precisely

$$X_t - X_s \sim \Gamma(\alpha^+(t-s), \lambda^+; \alpha^-(t-s), \lambda^-) \quad \text{for } 0 \leq s < t. \tag{3.2}$$

The characteristic triplet with respect to the truncation function $h \equiv 0$ is given by $(0, 0, F)$, where F denotes the Lévy measure

$$F(dx) = \left(\frac{\alpha^+}{x} e^{-\lambda^+ x} \mathbb{1}_{(0,\infty)}(x) + \frac{\alpha^-}{|x|} e^{-\lambda^- |x|} \mathbb{1}_{(-\infty,0)}(x) \right) dx. \tag{3.3}$$

The *cumulant generating function* $\Psi(z) = \ln \mathbb{E}[e^{zX_1}]$ exists on $(-\lambda^-, \lambda^+)$ and is given by

$$\Psi(z) = \alpha^+ \ln \left(\frac{\lambda^+}{\lambda^+ - z} \right) + \alpha^- \ln \left(\frac{\lambda^-}{\lambda^- + z} \right), \quad z \in (-\lambda^-, \lambda^+). \tag{3.4}$$

We can write $X = X^+ - X^-$ as the difference of two independent standard Gamma processes, where $X^+ \sim \Gamma(\alpha^+, \lambda^+)$ and $X^- \sim \Gamma(\alpha^-, \lambda^-)$. The corresponding cumulant generating functions $\Psi^+(z) = \ln \mathbb{E}[e^{zX_1^+}]$ and $\Psi^-(z) = \ln \mathbb{E}[e^{zX_1^-}]$ are given by

$$\Psi^+(z) = \alpha^+ \ln \left(\frac{\lambda^+}{\lambda^+ - z} \right), \quad z \in (-\infty, \lambda^+) \tag{3.5}$$

$$\Psi^-(z) = \alpha^- \ln \left(\frac{\lambda^-}{\lambda^- - z} \right), \quad z \in (-\infty, \lambda^-). \tag{3.6}$$

Note that $\Psi(z) = \Psi^+(z) + \Psi^-(-z)$ for $z \in (-\lambda^-, \lambda^+)$.

A *bilateral Gamma stock model* is an exponential Lévy model of the type (1.1) with X being a bilateral Gamma process. In what follows, we assume that $r \geq q \geq 0$, that is, the dividend rate q of the stock cannot exceed the interest rate r of the bank account and none of them is negative.

3.1. **Lemma.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$.*

(1) *If $\lambda^+ > 1$, then $\mathbb{P}$ is a martingale measure if and only if*

$$\left(\frac{\lambda^+}{\lambda^+ - 1} \right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + 1} \right)^{\alpha^-} = e^{r-q}. \tag{3.7}$$

(2) *If $\lambda^+ \leq 1$, then $\mathbb{P}$ is not a martingale measure.*

Proof. If $\lambda^+ > 1$, we have $\mathbb{E}[e^{X_1}] < \infty$. By Lemma 2.6 the discounted price process $\tilde{S}$ in (1.2) is a local martingale if and only if $\mathbb{E}[e^{X_1 - (r-q)}] = 1$, which is the case if and only if (3.7) holds.

In the case $\lambda^+ \leq 1$ we have $\mathbb{E}[e^{X_1}] = \infty$. Lemma 2.6 implies that $\tilde{S}$ cannot be a local martingale. □

4. Existence of Esscher martingale measures in bilateral Gamma stock models

For option pricing in bilateral Gamma stock models of the type (1.1) we have to find a martingale measure. One method is to use the so-called *Esscher transform*, which was pioneered in [10]. We recall the definition in the context of bilateral Gamma processes.

4.1. **Definition.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$ and let $\Theta \in (-\lambda^-, \lambda^+)$ be arbitrary. The* Esscher transform $\mathbb{P}^\Theta$ *is defined as the locally equivalent probability measure with likelihood process*

$$\Lambda_t(\mathbb{P}^\Theta, \mathbb{P}) := \frac{d\mathbb{P}^\Theta}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = e^{\Theta X_t - \Psi(\Theta) t}, \quad t \geq 0$$

where Ψ denotes the cumulant generating function given by (3.4).

4.2. **Lemma.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$ and let $\Theta \in (-\lambda^-, \lambda^+)$ be arbitrary. Then we have $X \sim \Gamma(\alpha^+, \lambda^+ - \Theta; \alpha^-, \lambda^- + \Theta)$ under $\mathbb{P}^\Theta$.*

Proof. This follows by combining Prop. 6.1, Prop. 6.3 and equation (6.9) from [18]. □

The upcoming result characterizes all bilateral Gamma stock models, for which martingale measures given by an Esscher transform exist.

4.3. **Theorem.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$. Then there exists $\Theta \in (-\lambda^-, \lambda^+)$ such that $\mathbb{P}^\Theta$ is a martingale measure if and only if*

$$\lambda^+ + \lambda^- > 1. \tag{4.1}$$

If (4.1) is satisfied, Θ is unique, belongs to the interval $(-\lambda^-, \lambda^+ - 1)$, and it is the unique solution of the equation

$$\left(\frac{\lambda^+ - \Theta}{\lambda^+ - \Theta - 1} \right)^{\alpha^+} \left(\frac{\lambda^- + \Theta}{\lambda^- + \Theta + 1} \right)^{\alpha^-} = e^{r-q}, \quad \Theta \in (-\lambda^-, \lambda^+ - 1). \tag{4.2}$$

Moreover, we have $X \sim \Gamma(\alpha^+, \lambda^+ - \Theta; \alpha^-, \lambda^- + \Theta)$ under $\mathbb{P}^\Theta$.

Proof. Let $\Theta \in (-\lambda^-, \lambda^+)$ be arbitrary. In view of Lemma 4.2 and Lemma 3.1, the probability measure $\mathbb{P}^\Theta$ is a martingale measure if and only if $\lambda^+ - \Theta > 1$, i.e. $\Theta \in (-\lambda^-, \lambda^+ - 1)$, and (4.2) is fulfilled. Note that $(-\lambda^-, \lambda^+ - 1) \neq \emptyset$ if and only if (4.1) is satisfied.

Provided (4.1), equation (4.2) is satisfied if and only if

$$f(\Theta) = r - q, \tag{4.3}$$

where $f : (-\lambda^-, \lambda^+ - 1) \to \mathbb{R}$ is defined as $f(\Theta) := f^+(\Theta) + f^-(\Theta)$ with

$$f^+(\Theta) := \alpha^+(\ln(\lambda^+ - \Theta) - \ln(\lambda^+ - 1 - \Theta)),$$
$$f^-(\Theta) := \alpha^-(\ln(\lambda^- + \Theta) - \ln(\lambda^- + 1 + \Theta)).$$

Taking derivatives of f^+ and f^-, we get that f is strictly increasing. Noting that

$$\lim_{\Theta \downarrow -\lambda^-} f(\Theta) = -\infty \quad \text{and} \quad \lim_{\Theta \uparrow \lambda^+ - 1} f(\Theta) = \infty,$$

there exists a unique $\Theta \in (-\lambda^-, \lambda^+ - 1)$ fulfilling (4.3). □

Hence, a martingale measure, which is an Esscher transform $\mathbb{P}^\Theta$, exists if and only if (4.1) is satisfied. In order to find the parameter $\Theta \in (-\lambda^-, \lambda^+ - 1)$, we have to solve equation (4.2). In general, this has to be done numerically. In the particular situation $\alpha^+ = \alpha^-$, which according to [18, Thm. 3.3] is the case if and only if X is Variance Gamma, and $r = q$ the solution for equation (4.2) is given by $\Theta = \frac{1}{2}(\lambda^+ - \lambda^- - 1)$, the midpoint of the interval $(-\lambda^-, \lambda^+ - 1)$.

5. Existence of minimal entropy martingale measures in bilateral Gamma stock models

We have seen in the previous section that in bilateral Gamma stock models an equivalent martingale measure is easy to obtain by solving equation (4.2), provided condition (4.1) is satisfied. Moreover, the driving process X is still a bilateral Gamma process under the new measure, which allows numerical calculations of option prices. However, it is not clear that, in reality, the market chooses this kind of measure.

In the literature, one often performs option pricing by finding the equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ which minimizes

$$\mathbb{E}[f(\Lambda_1(\mathbb{Q}, \mathbb{P}))]$$

for a strictly convex function $f : (0, \infty) \to \mathbb{R}$. Examples for the functional f are $f(x) = x^q$ for some $q > 1$ (see, e.g., [14, 2]) and $f(x) = x \ln x$. A popular choice, which we consider in this section, is the functional $f(x) = x \ln x$. Then, the quantity

$$\mathbb{H}(\mathbb{Q} \,|\, \mathbb{P}) := \mathbb{E}[\Lambda_1(\mathbb{Q}, \mathbb{P}) \ln \Lambda_1(\mathbb{Q}, \mathbb{P})] = \mathbb{E}_{\mathbb{Q}}[\ln \Lambda_1(\mathbb{Q}, \mathbb{P})]$$

is called the *relative entropy*. In connection with exponential Lévy models, it has been studied, e.g., in [4, 11, 9, 12], see also [17]. The minimal entropy has an information theoretic interpretation: minimizing relative entropy corresponds to choosing a martingale measure by adding the least amount of information to the prior model.

From a mathematical point of view, minimizing the relative entropy is convenient, because there is a connection between the minimal entropy and Esscher transforms of the exponential transform $\tilde{Y} := \mathcal{L}(\tilde{S})$ of the Lévy process $Y_t := X_t - (r-q)t$, which has rigorously been presented in [9] and further been extended in [12].

We shall now recall this connection in our framework, where X is a bilateral Gamma process. By [12, Thm. 2], the exponential transform $\tilde{Y}$ is a Lévy process, which has the characteristic triplet

$$\tilde{b} = \int_{-1}^{1} (e^x - 1) F(dx) - (r - q),$$
$$\tilde{c} = 0,$$
$$\tilde{F}(B) = \int_{\mathbb{R}} \mathbb{1}_B (e^x - 1) F(dx), \quad B \in \mathcal{B}(\mathbb{R}).$$

with respect to the truncation function $h(x) = x\mathbb{1}_{[-1,1]}(x)$, where the Lévy measure F is given by (3.3). Therefore, the cumulant generating function $\tilde{\Psi}$ of $\tilde{Y}$ exists on $\mathbb{R}_- := (-\infty, 0]$ and is given by

$$\begin{aligned} \tilde{\Psi}(z) &= -(r-q)z + \int_{\mathbb{R}} (e^{zx} - 1) \tilde{F}(dx) \\ &= -(r-q)z + \int_{\mathbb{R}} \left(e^{z(e^x - 1)} - 1 \right) F(dx), \quad z \le 0. \end{aligned} \tag{5.1}$$

5.1. **Definition.** *Let $\vartheta \le 0$ be arbitrary. The* Esscher transform $\mathbb{P}_\vartheta$ *is the locally equivalent probability measure with likelihood process*

$$\Lambda_t(\mathbb{P}_\vartheta, \mathbb{P}) := \frac{d\mathbb{P}_\vartheta}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{\vartheta \tilde{Y}_t - \tilde{\Psi}(\vartheta) t}, \quad t \ge 0.$$

The following result, which establishes the connection between the minimal entropy and Esscher transforms, will be crucial for our investigations.

5.2. **Proposition.** *There exists a minimal entropy martingale measure if and only if there exists $\vartheta \le 0$ such that $\mathbb{E}_\vartheta[\tilde{Y}_1] = 0$. In this case, one minimal entropy martingale measure is given by $\mathbb{P}_\vartheta$.*

Proof. The assertion follows from [12, Thm. 8] and Lemma 2.5. □

5.3. **Lemma.** [18, Lemma 6.2] *For all $\lambda_1, \lambda_2 > 0$ we have*

$$\int_0^\infty \frac{e^{-\lambda_2 x} - e^{-\lambda_1 x}}{x} dx = \ln\left(\frac{\lambda_1}{\lambda_2}\right).$$

We are now ready to characterize all bilateral Gamma stock models, for which minimal entropy martingale measures exist, and to determine the value of this minimal entropy.

5.4. **Theorem.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$.*

(1) *If $\lambda^+ \le 1$, there exists a unique $\vartheta < 0$ such that $\mathbb{P}_\vartheta$ is a minimal entropy martingale measure. It is the unique solution of the equation*

$$\begin{aligned} &\alpha^+ \int_0^\infty \frac{1}{x} e^{-\lambda^+ x} (e^x - 1) e^{\vartheta(e^x - 1)} dx \\ &+ \alpha^- \int_0^\infty \frac{1}{x} e^{-\lambda^- x} (e^{-x} - 1) e^{\vartheta(e^{-x} - 1)} dx = r - q, \quad \vartheta \in (-\infty, 0). \end{aligned} \tag{5.2}$$

(2) *If $\lambda^+ > 1$, there exists $\vartheta \le 0$ such that $\mathbb{P}_\vartheta$ is a minimal entropy measure if and only if*

$$\alpha^+ \ln\left(\frac{\lambda^+}{\lambda^+ - 1}\right) + \alpha^- \ln\left(\frac{\lambda^-}{\lambda^- + 1}\right) \ge r - q. \tag{5.3}$$

If (5.3) is satisfied, ϑ is unique and it is the unique solution of equation (5.2) for $\vartheta \in \mathbb{R}_-$.

(3) *If a minimal entropy martingale measure exists, then the value of the minimal entropy is given by*

$$\begin{aligned}\mathbb{H}(\mathbb{P}_\vartheta \,|\, \mathbb{P}) = &-\alpha^+ \int_0^\infty \frac{1}{x} e^{-\lambda^+ x} (e^{\vartheta(e^x - 1)} - 1) dx \\ &- \alpha^- \int_0^\infty \frac{1}{x} e^{-\lambda^- x} (e^{\vartheta(e^{-x} - 1)} - 1) dx + (r - q)\vartheta.\end{aligned} \tag{5.4}$$

Proof. For each $\vartheta \leq 0$ the characteristic triplet of $\tilde{Y}$ with respect to the truncation function $h(x) = x\mathbb{1}_{[-1,1]}(x)$ under the measure $\mathbb{P}_\vartheta$ is, according to [12, Thm. 1], given by

$$\begin{aligned}\tilde{b}_\vartheta &= \tilde{b} + \int_{\mathbb{R}} (e^{\vartheta x} - 1) h(x) \tilde{F}(dx) = \int_{-1}^1 (e^x - 1) e^{\vartheta(e^x - 1)} F(dx) - (r - q), \\ \tilde{c}_\vartheta &= 0, \\ \tilde{F}_\vartheta(dx) &= e^{\vartheta x} \tilde{F}(dx).\end{aligned}$$

Hence, we have $\mathbb{E}_\vartheta|\tilde{Y}_1| < \infty$, $\vartheta < 0$ and, by (3.3), the expectations are given by

$$\begin{aligned}\mathbb{E}_\vartheta[\tilde{Y}_1] &= \int_{\mathbb{R}} (e^x - 1) e^{\vartheta(e^x - 1)} F(dx) - (r - q) \\ &= \alpha^+ \int_0^\infty \frac{1}{x} e^{-\lambda^+ x} (e^x - 1) e^{\vartheta(e^x - 1)} dx \\ &\quad + \alpha^- \int_0^\infty \frac{1}{x} e^{-\lambda^- x} (e^{-x} - 1) e^{\vartheta(e^{-x} - 1)} dx - (r - q), \quad \vartheta \leq 0.\end{aligned}$$

Moreover, we have $\mathbb{E}_0|\tilde{Y}_1| < \infty$ if and only if $\lambda^+ > 1$, and in this case the expectation is, by Lemma 5.3, given by

$$\mathbb{E}_0[\tilde{Y}_1] = \alpha^+ \ln\left(\frac{\lambda^+}{\lambda^+ - 1}\right) + \alpha^- \ln\left(\frac{\lambda^-}{\lambda^- + 1}\right) - (r - q) \tag{5.5}$$

According to [24, Lemma 26.4] the cumulant generating function $\tilde{\Psi}$ of $\tilde{Y}$ is of class C^∞ on $(-\infty, 0)$, we have $\tilde{\Psi}'' > 0$ on $(-\infty, 0)$ and the first derivative is, by the representation (5.1), given by

$$\tilde{\Psi}'(\vartheta) = -(r - q) + \int_{\mathbb{R}} (e^x - 1) e^{\vartheta(e^x - 1)} F(dx) = \mathbb{E}_\vartheta[\tilde{Y}_1], \quad \vartheta \in (-\infty, 0).$$

Note that $\tilde{\Psi}'(\vartheta) \downarrow -\infty$ for $\vartheta \downarrow -\infty$.

Let us now consider the situation $\lambda^+ \leq 1$. Then we have $\tilde{\Psi}'(\vartheta) \uparrow \infty$ for $\vartheta \uparrow 0$. Since $\tilde{\Psi}'$ is continuous and $\tilde{\Psi}'' > 0$ on $(-\infty, 0)$, Proposition 5.2 yields the first statement.

In the case $\lambda^+ > 1$, an analogous argumentation yields, by taking into account (5.5), the second statement.

If a minimal entropy martingale measure exists, then the value of the minimal entropy is given by

$$\mathbb{H}(\mathbb{P}_\vartheta \,|\, \mathbb{P}) = \mathbb{E}_\vartheta[\ln \Lambda_1(\mathbb{P}_\vartheta, \mathbb{P})] = \mathbb{E}_\vartheta[\vartheta \tilde{Y}_1 - \tilde{\Psi}(\vartheta)] = -\tilde{\Psi}(\vartheta),$$

which, by (5.1) and (3.3), gives us (5.4). □

We emphasize that for bilateral Gamma stock models the value of the minimal entropy in (5.4) can easily be calculated numerically in terms of the parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^-$.

6. Existence of minimal entropy measures preserving the class of bilateral Gamma processes

We have seen in the previous section that for bilateral Gamma stock models we obtain the minimal entropy martingale measure $\mathbb{P}_\vartheta$ by solving equation (5.2) numerically, provided $\lambda^+ \leq 1$ or condition (5.3) is satisfied. Under $\mathbb{P}_\vartheta$, the bilateral Gamma process X is still a Lévy process, a result which is due to [9], and we know its characteristic triplet. However, its characteristic function is not available in closed form, whence we cannot perform option pricing by the method of Fourier transformation.

Recall that, on the other hand, the Esscher transform from Section 4 leaves the family of bilateral Gamma processes invariant.

Our idea in this section is therefore as follows. We minimize the relative entropy within the class of bilateral Gamma processes by performing *bilateral* Esscher transforms.

6.1. **Definition.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$ and let $\theta^+ \in (-\infty, \lambda^+)$ and $\theta^- \in (-\infty, \lambda^-)$ be arbitrary. The* bilateral Esscher transform $\mathbb{P}^{(\theta^+,\theta^-)}$ *is defined as the locally equivalent probability measure with likelihood process*

$$\Lambda_t(\mathbb{P}^{(\theta^+,\theta^-)}, \mathbb{P}) := \frac{d\mathbb{P}^{(\theta^+,\theta^-)}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{\theta^+ X_t^+ - \Psi^+(\theta^+)t} \cdot e^{\theta^- X_t^- - \Psi^-(\theta^-)t}, \quad t \geq 0$$

where Ψ^+, Ψ^- denote the cumulant generating functions given by (3.5), (3.6).

Note that the Esscher transforms $\mathbb{P}^\Theta$ from Section 4 are special cases of the just introduced bilateral Esscher transforms $\mathbb{P}^{(\theta^+,\theta^-)}$. Indeed, it holds

$$\mathbb{P}^\Theta = \mathbb{P}^{(\Theta,-\Theta)}, \quad \Theta \in (-\lambda^-, \lambda^+). \tag{6.1}$$

6.2. **Lemma.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$ and let $\theta^+ \in (-\infty, \lambda^+)$ and $\theta^- \in (-\infty, \lambda^-)$ be arbitrary. Then we have $X \sim \Gamma(\alpha^+, \lambda^+ - \theta^+; \alpha^-, \lambda^- - \theta^-)$ under $\mathbb{P}^{(\theta^+,\theta^-)}$.*

Proof. This follows by combining Prop. 6.1, Prop. 6.3 and equation (6.9) from [18]. □

6.3. **Lemma.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$ and let $\theta^+ \in (-\infty, \lambda^+)$ and $\theta^- \in (-\infty, \lambda^-)$ be arbitrary. Then $\mathbb{P}^{(\theta^+,\theta^-)}$ is a martingale measure if and only if $\theta^+ \in (\lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1}, \lambda^+ - 1)$ and*

$$\theta^- = \Phi(\theta^+), \tag{6.2}$$

where $\Phi : (\lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1}, \lambda^+ - 1) \to (-\infty, \lambda^-)$ is defined as

$$\Phi(\theta) := \lambda^- - \left(\sqrt[\alpha^-]{\left(\frac{\lambda^+ - \theta}{\lambda^+ - \theta - 1} \right)^{\alpha^+} e^{-(r-q)} - 1} \right)^{-1}. \tag{6.3}$$

By convention, we set $\lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1} := -\infty$ if $r = q$.

Proof. This is an immediate consequence of Lemma 6.2 and Lemma 3.1. □

Introducing the set of parameters

$$\mathcal{M}_\mathbb{P} := \{(\theta^+, \theta^-) \in (-\infty, \lambda^+) \times (-\infty, \lambda^-) \,|\, \mathbb{P}^{(\theta^+,\theta^-)} \text{ is a martingale measure}\}$$

such that the bilateral Esscher transform is a martingale measure, the previous Lemma 6.3 tells us that

$$\mathcal{M}_\mathbb{P} = \{(\theta, \Phi(\theta)) \in \mathbb{R}^2 \,|\, \theta \in (\lambda^+ - (1 - \exp(-\tfrac{r-q}{\alpha^+}))^{-1}, \lambda^+ - 1)\}. \tag{6.4}$$

The following consequence contributes to Theorem 4.3. It tells us that, provided (4.1) is satisfied, we can, instead of solving (4.2), alternatively solve equation (6.5) below in order to find the Esscher transform $\mathbb{P}^{\Theta}$.

6.4. **Corollary.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$. If (4.1) is satisfied, then the unique $\Theta \in (-\lambda^-, \lambda^+ - 1)$ such that $\mathbb{P}^{\Theta}$ is a martingale measure is the unique solution of the equation*

$$\Phi(\Theta) = -\Theta, \quad \Theta \in (\lambda^+ - (1 - \exp(-\tfrac{r-q}{\alpha^+}))^{-1}, \lambda^+ - 1). \tag{6.5}$$

Proof. The proof follows from Theorem 4.3 and relations (6.1), (6.4). □

We have considered the case $\alpha^+ = \alpha^-$ and $r = q$ at the end of Section 4. In this particular situation $\Phi : (-\infty, \lambda^+ - 1) \to (-\infty, \lambda^-)$ is given by $\Phi(\theta) = \lambda^- - \lambda^+ + \theta + 1$, whence we see again, this time by solving equation (6.5), that the Esscher parameter is given by $\Theta = \frac{1}{2}(\lambda^+ - \lambda^- - 1)$.

We are now ready to treat the existence of minimal entropy martingale measures in bilateral Gamma stock markets, under which X remains a bilateral Gamma process.

6.5. **Theorem.** *Let $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under $\mathbb{P}$ with arbitrary parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^- > 0$. Then, there exist $\theta^+ \in (-\infty, \lambda^+)$ and $\theta^- \in (-\infty, \lambda^-)$ such that*

$$\mathbb{H}(\mathbb{P}^{(\theta^+,\theta^-)} \,|\, \mathbb{P}) = \min_{(\vartheta^+,\vartheta^-) \in \mathcal{M}_{\mathbb{P}}} \mathbb{H}(\mathbb{P}^{(\vartheta^+,\vartheta^-)} \,|\, \mathbb{P}). \tag{6.6}$$

In this case, we have $\theta^+ \in (\lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1}, \lambda^+ - 1)$, relation (6.2) is satisfied, and θ^+ minimizes the function $f : (\lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1}, \lambda^+ - 1) \to \mathbb{R}$ defined as

$$\begin{aligned} f(\theta) &:= \alpha^+ \left(\frac{\lambda^+}{\lambda^+ - \theta} - 1 - \ln \left(\frac{\lambda^+}{\lambda^+ - \theta} \right) \right) \\ &\quad + \alpha^- \left(\frac{\lambda^-}{\lambda^- - \Phi(\theta)} - 1 - \ln \left(\frac{\lambda^-}{\lambda^- - \Phi(\theta)} \right) \right). \end{aligned} \tag{6.7}$$

Moreover, we have $X \sim \Gamma(\alpha^+, \lambda^+ - \theta^+; \alpha^-, \lambda^- - \theta^-)$ under $\mathbb{P}^{(\theta^+,\theta^-)}$, and the value of the minimal entropy is given by

$$\mathbb{H}(\mathbb{P}^{(\theta^+,\theta^-)} \,|\, \mathbb{P}) = f(\theta^+). \tag{6.8}$$

Proof. By [18, Prop. 6.3], for all $(\theta^+, \theta^-) \in \mathcal{M}_{\mathbb{P}}$ the relative entropy $\mathbb{H}(\mathbb{P}^{(\theta^+,\theta^-)} \,|\, \mathbb{P})$ is given by (6.8). We can write the function f as

$$f(\theta) = \alpha^+ g\left(\frac{\lambda^+}{\lambda^+ - \theta} \right) + \alpha^- g\left(\frac{\lambda^-}{\lambda^- - \Phi(\theta)} \right), \tag{6.9}$$

where $g : (0, \infty) \to \mathbb{R}$ denotes the strictly convex function $g(x) = x - 1 - \ln x$. Note that

$$\lim_{\theta \downarrow \lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1}} \frac{\lambda^+}{\lambda^+ - \theta} \in [0, \lambda^+) \quad \text{and} \quad \lim_{\theta \uparrow \lambda^+ - 1} \frac{\lambda^+}{\lambda^+ - \theta} = \lambda^+$$

as well as

$$\lim_{\theta \downarrow \lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1}} \frac{\lambda^-}{\lambda^- - \Phi(\theta)} = 0 \quad \text{and} \quad \lim_{\theta \uparrow \lambda^+ - 1} \frac{\lambda^-}{\lambda^- - \Phi(\theta)} = \infty.$$

We conclude that

$$\lim_{\theta \downarrow \lambda^+ - (1 - \exp(-\frac{r-q}{\alpha^+}))^{-1}} f(\theta) = \infty \quad \text{and} \quad \lim_{\theta \uparrow \lambda^+ - 1} f(\theta) = \infty.$$

Since f is continuous, it attains a minimum and the assertion follows. □

Consequently, unlike the Esscher transform $\mathbb{P}^\theta$ from Section 4 and the real minimal entropy martingale measure $\mathbb{P}_\vartheta$ from Section 5, the minimal entropy martingale measure $\mathbb{P}^{(\theta,\Phi(\theta))}$ in the subclass of all measures which leave bilateral Gamma processes invariant, *always* exists. Also in this case it is possible to determine the minimal entropy numerically by minimizing the function f defined in (6.7). A comparison with the value in (5.4) will be given in a concrete example in Section 7.

6.6. **Remark.** *If condition (4.1) is satisfied, choosing the Esscher parameter* $\Theta \in (-\lambda^-, \lambda^+ - 1)$ *from Section 4 and inserting it into the function* f*, we obtain, by Corollary 6.4 and the representation (6.9) of* f*,*

$$\mathbb{H}(\mathbb{P}^\Theta \,|\, \mathbb{P}) = f(\Theta) = \alpha^+ g\bigg(\frac{\lambda^+}{\lambda^+ - \Theta}\bigg) + \alpha^- g\bigg(\frac{\lambda^-}{\lambda^- + \Theta}\bigg). \tag{6.10}$$

Note that one of the arguments in (6.10) for the function g *is greater than* 1*, whereas the other argument is smaller than* 1*. Since the strictly convex function* g *attains its global minimum at* $x = 1$*, the relative entropy* $\mathbb{H}(\mathbb{P}^\Theta \,|\, \mathbb{P})$ *in (6.10) is, in general, not too far from the minimal relative entropy* $\mathbb{H}(\mathbb{P}^{(\theta,\Phi(\theta))} \,|\, \mathbb{P})$ *in (6.6). Therefore, we expect that, typically, the Esscher parameter* Θ *and bilateral Esscher parameter* θ *are close to each other.*

In general, we have the inequalities

$$\mathbb{H}(\mathbb{P}_\vartheta \,|\, \mathbb{P}) < \mathbb{H}(\mathbb{P}^{(\theta,\Phi(\theta))} \,|\, \mathbb{P}) < \mathbb{H}(\mathbb{P}^\Theta \,|\, \mathbb{P}),$$

provided, the respective measures exist. Using our preceding results, we can compute the values of the respective entropies numerically and see, how close they are to each other.

Since X is still a bilateral Gamma process under the bilateral Esscher transform $\mathbb{P}^{(\theta,\Phi(\theta))}$, we can perform option pricing by the method of Fourier transformation.

The characteristic function of a bilateral Gamma distribution is given by (3.1) and all increments of X are bilateral Gamma distributed, see (3.2). Therefore, if $\lambda^+ - \theta > 1$, the price of a European call option with strike price K and time of maturity T is given by

(6.11)

$$\Pi = -\frac{e^{-rT}K}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} \left(\frac{K}{S_0}\right)^{iz} \left(\frac{\lambda^+ - \theta}{\lambda^+ - \theta + iz}\right)^{\alpha^+ T} \left(\frac{\lambda^- - \Phi(\theta)}{\lambda^- - \Phi(\theta) - iz}\right)^{\alpha^- T} \frac{dz}{z(z-i)},$$

where $\nu \in (1, \lambda^+ - \theta)$ is arbitrary. This follows from [20, Thm. 3.2].

7. A numerical illustration

We conclude this article with a numerical illustration of the preceding results. For a bilateral Gamma stock model of the type (1.1) we have estimated the parameters of the bilateral Gamma process X as

$$(\alpha^+, \lambda^+; \alpha^-, \lambda^-) = (1.55, 133.96; 0.94, 88.92)$$

from observations of the German stock index DAX, see [18, Sec. 9]. For our upcoming calculations, we take the initial stock price $S_0 = 5000$ and, for simplicity, we put $r = q = 0$.

We start with the computation of the Esscher transform from Section 4. Note that condition (4.1) is fulfilled. Solving equation (4.2), we obtain the Esscher transform $\mathbb{P}^\Theta$ with $\Theta = -5.28$. Under the measure $\mathbb{P}^\Theta$, the driving process X is bilateral Gamma with parameters

$$(\alpha^+, \lambda^+ - \Theta; \alpha^-, \lambda^- + \Theta) = (1.55, 139.24; 0.94, 83.64).$$

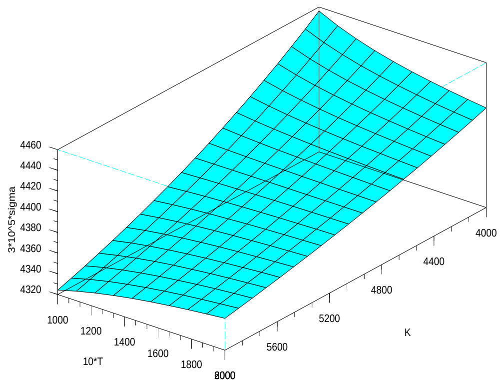

FIGURE 1. Implied volatility surface.

The value of the relative entropy, provided by (6.10), is given by

$$\mathbb{H}(\mathbb{P}^{\Theta} \,|\, \mathbb{P}) = 0.00294113. \tag{7.1}$$

We proceed with the minimal entropy martingale measure from Section 5. We have

$$\alpha^+ \ln\left(\frac{\lambda^+}{\lambda^+ - 1}\right) + \alpha^- \ln\left(\frac{\lambda^-}{\lambda^- + 1}\right) = 0.0000221, \tag{7.2}$$

whence condition (5.3) is satisfied. Solving (5.2), we obtain the minimal entropy martingale measure $\mathbb{P}_\vartheta$ with $\vartheta = -5.30$. Using (5.4), the value of the minimal entropy is given by

$$\mathbb{H}(\mathbb{P}_\vartheta \,|\, \mathbb{P}) = 0.00294091. \tag{7.3}$$

Finally, we turn to the computation of the bilateral Esscher transform from Section 6. Minimizing the function f given by (6.7), we obtain the bilateral Esscher transform $\mathbb{P}^{(\theta,\Phi(\theta))}$ with $\theta = -5.34$. We observe that the Esscher parameter Θ and the bilateral Esscher parameter θ are quite close to each other, see Remark 6.6. Under the measure $\mathbb{P}^{(\theta,\Phi(\theta))}$, the driving process X is bilateral Gamma with parameters

$$(\alpha^+, \lambda^+ - \theta; \alpha^-, \lambda^- - \Phi(\theta)) = (1.55, 139.30; 0.94, 83.68).$$

Computing the relative entropy according to (6.8) yields

$$\mathbb{H}(\mathbb{P}^{(\theta,\Phi(\theta))} \,|\, \mathbb{P}) = 0.00294107. \tag{7.4}$$

The relative entropies computed in (7.1), (7.3), (7.4) show that both, the Esscher transform $\mathbb{P}^{\Theta}$ and the bilateral Esscher transform $\mathbb{P}^{(\theta,\Phi(\theta))}$ are quite close to the minimal entropy martingale measure $\mathbb{P}_\vartheta$.

Under the martingale measure $\mathbb{P}^{(\theta,\Phi(\theta))}$, we can numerically compute the prices of European call options by using formula (6.11). Figure 1 shows the implied volatility surface. We observe the following properties:

- The dependence of the implied volatility with respect to the strike price K is decreasing, we have a so-called "skew".
- The skew flattens out for large times of maturity T.

Hence, the implied volatility surface in Figure 1 has the typical features, which one observes in practice.

7.1. **Remark.** *The basic tool for all calculations in this paper is the cumulant generating function (3.4) of the bilateral Gamma distribution. There exists a more general class of infinitely divisible distributions, the so-called tempered stable distributions (see* [5, Sec. 4.5]*). They have a comparatively simple cumulant generating function, too, and thus essential parts of the present paper can also be proven for them. A disadvantage is the lack of an explicit form of the density, which does not allow to apply maximum likelihood estimators in order to determine the parameters from observations of a stock. The concrete results will be published elsewhere.*

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