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JUMP-DIFFUSIONS IN HILBERT SPACES: EXISTENCE, STABILITY AND NUMERICS

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Abstract. By means of an original approach, called "method of the moving frame", we establish existence, uniqueness and stability results for mild and weak solutions of stochastic partial differential equations (SPDEs) with path dependent coefficients driven by an infinite dimensional Wiener process and a compensated Poisson random measure. Our approach is based on a time-dependent coordinate transform, which reduces a wide class of SPDEs to a class of simpler SDE problems. We try to present the most general results, which we can obtain in our setting, within a self-contained framework to demonstrate our approach in all details. Also an outlook towards a general theory of numerical approaches to SPDEs is provided in the spirit of our setting.

Key Words: stochastic partial differential equations, mild and weak solutions, stability results, high-order numerical schemes.

60H15, 60H35

1. Introduction

Stochastic partial differential equations (SPDEs) are usually considered as stochastic perturbations of partial differential equations (PDEs). More precisely, let H be a Hilbert space and A the generator of a strongly continuous semigroup S on H, then

$$\frac{dr_t}{dt} = Ar_t + \alpha(r_t), \quad r_0 \in H$$

describes a (semi-linear) PDE on the Hilbert space of states H with linear generator A and (non-linear) term $\alpha : H \to H$. Solutions are usually defined in the mild or weak sense. A stochastic perturbation of this (semi-linear) PDE is given through a driving noise and (volatility) vector fields, for instance one can choose a one-dimensional Brownian motion W and $\sigma : H \to H$ and consider

$$dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t, \quad r_0 \in H.$$

Solution concepts, properties of solutions, manifold applications have been worked out in the most general cases, e.g., [9] in the case of Brownian noise or [24] in the case of Lévy noises.

We suggest in this article a new approach to SPDEs, which works for most of the SPDEs considered in the literature (namely those where the semigroup is pseudo-contractive). The advantages are three-fold: first one can consider most general noises with path-dependent coefficients and derive existence, uniqueness and stability results in an easy manner. Second the new approach easily leads to (numerical) approximation schemes for SPDEs, third the approach allows for rough path formulations (see [30]) and therefore for large deviation results, Freidlin-Wentzell

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type results, etc. In this article we shall mainly address existence, uniqueness and stability results for SPDEs with driving Poisson random measures and general path-dependent coefficients. An outline of the basic relation, namely short-time asymptotics, for high-order, weak or stong numerical schemes is presented, too.

In our point of view SPDEs are considered as time-dependent transformations of well-understood stochastics differential equations (SDEs). This is best described by a metapher from physics: take the previous equation and assume $\dim H = 1$, $\alpha(r) = 0$ and $\sigma(r) = \sigma$ a constant, i.e. an Ornstein-Uhlenbeck process

$$dr_t = Ar_t dt + \sigma dW_t,\ r_0 \in \mathbb{R}$$

in dimension one describing the trajectory of a Brownian particle in a (dumping) velocity field $x \mapsto Ax$. If we move our coordinate frame according to the vector field $x \mapsto Ax$ we observe a transformed movement of the particle, namely

$$df_t = \exp(-At)\sigma dB_t,\ f_0 = r_0,$$

which corresponds to a Brownian motion with time-dependent volatility, since space is scaled by a factor $\exp(-At)$ at time t and the speed of the movement of the coordinate frame makes the drift disappear. Loosely speaking, one "jumps on the moving frame", where the speed of the frame is chosen equal to the drift. In finite dimensions the advantage of this procedure is purely conceptual, since analytically the both equations can be equally well treated. If one imagines for a moment the same procedure for an SPDE the advantage is much more than conceptual, since the transformed equation, seen from the moving frame, is rather an SDE than an SPDE, since the non-continuous drift term disappears in the moving frame. At this point it is clear that the drift term in infinite dimensions does usually not allow movements in negative time direction, which is crucial for the approach. This limitation can be overcome by the Skőkefalvi-Nagy theorem, which allows for group extensions of given (pseudo-contractive) semigroups of linear operators. We emphasize that we do not need the particular structure of this extension, which might be quite involved.

Therefore we suggest the following approach to SPDEs, which is the guideline through this article:

- consider the SDE obtained by transforming the SPDE with a time-dependent transformation $r \mapsto S_{-t}r$ (jump to the moving frame).
- solve the transformed SDE.
- transform the solution process by $r \mapsto S_t r$ in order to obtain a mild solution of the original SPDE (leave the moving frame).

The emphasis of this article to provide a self-contained outline of this method in the realm of jump-diffusions with path-dependent coefficients, which has not been treated in the literature so far.

In [1] and [23] existence, uniqueness and regular dependence on initial data are considered for SPDEs driven by a Wiener processes and Poisson random measures. The authors also apply the Skőkefalvi-Nagy theorem to prove certain inequalities, which are crucial for their considerations. In contrast our approach means that we reduce all these separate considerations to the analysis of *one* transformed SDE, which corresponds then – by means of the time-dependent transformation – to the solution of the given SPDE.

Our approach is based on the general jump-diffusion approach to stochastic partial differential equations as presented in [18] or [23]. In contrast, our vector fields can can be path-dependent in a general sense, not only random as supposed in [23]. Applications of this setting can be found in recent work on volatility surfaces, where random dependence of the vector fields is not enough. We first do the obvious proofs for stochastic differential equations with values in (separable) Hilbert spaces.

Then we show that by our transformation method ("jump to the moving frame") we can transfer those results to stochastic partial differential equations. In a completely similar way we could have taken the setting for stochastic differential equations in Ph. Protter's book [25], which is based on semi-martingales as driving processes and where we can litterally transfer the respective theorems into the setting of stochastic partial differential equations. In particular all L^p-estimates – as extensively proved in [25] – can be transfered into the setting of stochastic partial differential equations.

The "moving frame approach" is a particular case of methods, where pull-backs with respect to flows are applied. Those methods have quite a long history in the theory of ODEs, PDEs and SDEs (pars pro toto we mention the Doss-Sussman method as described in [26] and the further material therein). In the realm of SPDEs the "pull-back" method has been successfully applied in [7] with respect to noise vector fields. See also a discussion in [6] where this point of view is applied again, but a pull-back with respect to the PDE part has not been applied yet.

We shall now provide a guideline for the remainder of the article. In Section 2 we define the fundamental concepts, notions and notations for stochastic integration with respect to Wiener processes and Poisson random measures. In Section 3 and 4 we provide for the sake of completeness existence and uniqueness results for Hilbert spaces valued SDEs and respective L^p-estimates. In Sections 5 and 6 we provide stability and regularity results for those SDEs. Section 7 we introduce all necessary solution concepts for (semi-linear) SPDEs. In Section 8 we apply our method of the moving frame to existence and uniqueness questions. Section 9 is devoted to the study of stability and regularity for SPDEs. Section 10 and Section 11 describe Markovian SPDE problems and several high order numerical schemes for SPDEs in this case. Again for the sake of completeness we provide a stochastic Fubini theorem with respect to compensated Poisson random measures in Appendix A.

2. Stochastic integration in Hilbert spaces

In this section, we shall outline the stochastic integrals with respect to an infinite dimensional Brownian motion and with respect to a compensated Poisson random measure.

2.1. **Setting and Definitions.** From now on, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Furthermore, let H denote a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and associated norm $\| \cdot \|_H$. If there is no ambiguity, we shall simply write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$.

In the sequel, $\mathcal{P}$ denotes the predictable σ-algebra on $\mathbb{R}_+$ and $\mathcal{P}_T$ denotes predictable σ-algebra on $[0,T]$ for an arbitrary $T \in \mathbb{R}_+$. We denote by λ the Lebesgue measure on $\mathbb{R}$.

For an arbitrary $p \geq 1$ and a finite time horizon $T \in \mathbb{R}_+$ we define

$$L_T^p(\lambda; H) := L^p(\Omega \times [0,T], \mathcal{P}_T, \mathbb{P} \otimes \lambda; H)$$

and let $L^p(\lambda; H)$ be the space of all predictable process $\Phi : \Omega \times \mathbb{R}_+ \to H$ such that for each $T \in \mathbb{R}_+$ the restriction of Φ to $\Omega \times [0,T]$ belongs to $L_T^p(\lambda; H)$. Furthermore, $L_{\mathrm{loc}}^p(\lambda; H)$ denotes the space of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \to H$ such that

$$\mathbb{P}\bigg(\int_0^T \| \Phi_t \|^p dt < \infty \bigg) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

Clearly, for each $\Phi \in L_{\mathrm{loc}}^p(\lambda; H)$ the path-by-path Stieltjes integral $\int_0^t \Phi_s ds$ exists.

Let $M_T^2(H)$ be the space of all square-integrable càdlàg martingales $M : \Omega \times [0,T] \to H$, where indistinguishable processes are identified. Endowed with the inner product

$$(M,N) \mapsto \mathbb{E}[\langle M_T, N_T \rangle],$$

the space $M_T^2(H)$ is a Hilbert space. The space $M_T^{2,c}(H)$, consisting of all continuous elements from $M_T^2(H)$, is a closed subspace of $M_T^2(H)$, which is a consequence of Doob's martingale inequality [9, Thm. 3.8].

2.2. **Stochastic Integration with respect to Wiener processes.** Let U be another separable Hilbert space and $Q \in \mathcal{L}(U)$ be a compact, self-adjoint, strictly positive linear operator. Then there exist an orthonormal basis $\{e_j\}$ of U and a bounded sequence λ_j of strictly positive real numbers such that

$$Qu = \sum_j \lambda_j \langle u, e_j \rangle e_j, \quad u \in U$$

namely, the λ_j are the eigenvalues of Q, and each e_j is an eigenvector corresponding to λ_j, see, e.g., [31, Thm. VI.3.2].

The space $U_0 := Q^{\frac{1}{2}}(U)$, equipped with inner product $\langle u, v \rangle_{U_0} := \langle Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v \rangle_U$, is another separable Hilbert space and $\{\sqrt{\lambda_j} e_j\}$ is an orthonormal basis.

Let W be a Q-Wiener process [9, p. 86,87]. We assume that $\operatorname{Tr} Q = \sum_j \lambda_j < \infty$. Otherwise, which is the case if W is a cylindrical Wiener process, there always exists a separable Hilbert space $U_1 \supset U$ on which W has a realization as a finite trace class Wiener process, see [9, Chap. 4.3].

We denote by $L_2^0 := L_2(U_0, H)$ the space of Hilbert-Schmidt operators from U_0 into H, which, endowed with the Hilbert-Schmidt norm

$$\|\Phi\|_{L_0^2} := \sqrt{\sum_j \lambda_j \|\Phi e_j\|^2}, \quad \Phi \in L_0^2$$

itself is a separable Hilbert space.

Following [9, Chap. 4.2], we define the stochastic integral $\int_0^t \Phi_s dW_s$ as an isometry, extending the obvious isometry on simple predictable processes, from $L_T^2(W; L_2^0)$ to $M_T^{2,c}(H)$, where

$$L_T^2(W; L_2^0) := L^2(\Omega \times [0,T], \mathcal{P}_T, \mathbb{P} \otimes \lambda; L_2^0).$$

In particular, we obtain the *Itô-isometry*

$$\mathbb{E}\left[\left\| \int_0^t \Phi_s dW_s \right\|^2 \right] = \mathbb{E}\left[\int_0^t \|\Phi_s\|_{L_2^0}^2 ds \right], \quad t \in [0,T] \tag{2.1}$$

for all $\Phi \in L_T^2(W; L_2^0)$. In a straightforward manner, we extend the stochastic integral to the space $L^2(W; L_2^0)$ of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \to L_0^2$ such that the restriction of Φ to $\Omega \times [0,T]$ belongs to $L_T^2(W; L_2^0)$ for all $T \in \mathbb{R}_+$, and, furthermore, to the space $L_{\mathrm{loc}}^2(W; L_2^0)$ consisting of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \to L_0^2$ such that

$$\mathbb{P}\bigg(\int_0^T \|\Phi_t\|_{L_2^0}^2 dt < \infty \bigg) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

There is an alternative view on the stochastic integral, which we shall use in this text. According to [9, Prop. 4.1], the sequence of stochastic processes $\{\beta^j\}$ defined as $\beta^j := \frac{1}{\sqrt{\lambda_j}} \langle W, e_j \rangle$ is a sequence of real-valued independent $(\mathcal{F}_t)$-Brownian motions and we have the expansion

$$W = \sum_j \sqrt{\lambda_j} \beta^j e_j,$$

where the series is convergent in the space $M^2(U)$ of U-valued square-integrable martingales. Let $\Phi \in L^2_{\mathrm{loc}}(W; L^0_2)$ be arbitrary. For each j we set $\Phi^j := \sqrt{\lambda_j}\Phi e_j$. Then we have

$$\int_0^t \Phi_s dW_s = \sum_j \int_0^t \Phi^j_s d\beta^j_s, \quad t \in \mathbb{R}_+ \tag{2.2}$$

where the convergence is uniformly on compact time intervals in probability, see [9, Thm. 4.3].

2.3. **Stochastic Integration with respect to Poisson random measures.** Let $(E, \mathcal{E})$ be a measurable space which we assume to be a *Blackwell space* (see [11, 15]). We remark that every Polish space with its Borel σ-field is a Blackwell space.

Now let μ be a homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [18, Def. II.1.20]. Then its compensator is of the form $dt \otimes F(dx)$, where F is a σ-finite measure on $(E, \mathcal{E})$.

We define the Itô-integral $\int_0^t \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds)$ as an isometry, which extends the obvious isometry on simple predictable processes, from $L^2_T(\mu; H)$ to $M^2_T(H)$, where

$$L^2_T(\mu; H) := L^2(\Omega \times [0,T] \times E, \mathcal{P}_T \otimes \mathcal{E}, \mathbb{P} \otimes \lambda \otimes F; H). \tag{2.3}$$

In particular, for each $\Phi \in L^2_T(\mu; H)$ we obtain the *Itô-isometry*

$$\mathbb{E}\left[\left\| \int_0^t \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds) \right\|^2\right] = \mathbb{E}\left[\int_0^t \int_E \|\Phi(s,x)\|^2 F(dx)ds\right] \tag{2.4}$$

for all $t \in [0,T]$. In a straightforward manner, we extend the stochastic integral to the space $L^2(\mu; H)$ of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \times E \to H$ such that the restriction of Φ to $\Omega \times [0,T] \times E$ belongs to $L^2_T(\mu; H)$ for all $T \in \mathbb{R}_+$, and, furthermore, to the space $L^2_{\mathrm{loc}}(\mu; H)$ consisting of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \times E \to H$ such that

$$\mathbb{P}\bigg(\int_0^T \|\Phi(t,x)\|^2 F(dx)dt < \infty \bigg) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

Such a construction of the stochastic integral can, e.g., be found in [2, Sec. 4] for the finite dimensional case and in [27], [20, Sec. 2] for the infinite dimensional case.

2.4. **Path properties of stochastic integrals.** It is apparent that for every $\Phi \in L^p_{\mathrm{loc}}(\lambda; H)$, where $p \geq 1$, the path-by-path Stieltjes integral $\int_0^\bullet \Phi_s ds$ has continuous sample paths.

As outlined in Section 2.2, we have first defined the stochastic integral $\int_0^t \Phi_s dW_s$ as an isometry from $L^2_T(W; L^0_2)$ to $M^{2,c}_T(H)$, the space of all square-integrable continuous martingales, and then extended it by localization. Therefore, for each $\Phi \in L^2_{\mathrm{loc}}(W; L^0_2)$, the trajectories of the integral process $\int_0^\bullet \Phi_s dW_s$ are continuous.

Similarly, the stochastic integral $\int_0^t \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds)$, outlined in Section 2.3, is, in the first step, defined as an isometry from $L^2_T(\mu; H)$ to $M^2_T(H)$, the space of all square-integrable càdlàg martingales, and then extended by localization. Hence, for each $\Phi \in L^2_{\mathrm{loc}}(\mu; H)$ the integral process $\int_0^\bullet \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds)$ has càdlàg sample paths.

3. Existence and uniqueness of solutions for stochastic differential equations

In this section we prove existence and uniqueness results for stochastic differential equations (SDEs) on a possibly infinite dimensional state space. The results are fairly standard, but we provide them in order to keep our presentation self-contained and to introduce certain notation which we shall need in the further sections.

For any $T \in \mathbb{R}_+$ the space $C([0,T]; L^2(\Omega; H))$ of all continuous functions from $[0,T]$ into $L^2(\Omega; H)$ is a Banach space with respect to the norm

$$\|r\|_{T,0} := \sup_{t\in[0,T]} \|r_t\|_{L^2(\Omega;H)} = \sqrt{\sup_{t\in[0,T]} \mathbb{E}[\|r_t\|^2]}.$$

Note that $C([0,T]; L^2(\Omega; H))$ is a space consisting of continuous curves of equivalence classes of random variables. Each element can be associated a process $r : \Omega \times \mathbb{R}_+ \to H$.

For each $a > 0$ an equivalent norm is given by

$$\|r\|_{T,a} := \sqrt{\sup_{t\in[0,T]} e^{-at}\mathbb{E}\left[\|r_t\|^2\right]}.$$

Since each adapted process $r \in C([0,T]; L^2(\Omega; H))$ has a predictable version according to [9, Prop. 3.6.ii], the subspace $C_{\text{pr}}[0,T] := C_{\text{pr}}([0,T]; L^2(\Omega; H))$ consisting of all predictable processes from $C([0,T]; L^2(\Omega; H))$ is closed with respect to this norm.

Let $C_{\text{pr}}(H)$ be the space of all processes $\Phi : \Omega\times\mathbb{R}_+ \to H$ such that the restriction of Φ to $\Omega \times [0,T]$ belongs to $C_{\text{pr}}[0,T]$ for all $T \in \mathbb{R}_+$.

We denote by $H_{\mathcal{P}}$ resp. $H_{\mathcal{P}\otimes\mathcal{E}}$ the space of all predictable processes $\Phi : \Omega\times\mathbb{R}_+ \to H$ resp. $\Phi : \Omega \times \mathbb{R}_+ \times E \to H$.

We shall now deal with stochastic differential equations of the kind

$$\left\{ \begin{array}{rcl} dr_t &=& \alpha(r)_t dt + \sigma(r)_t dW_t + \int_E \gamma(r)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &=& h_0, \end{array} \right. \tag{3.1}$$

where $\alpha : C_{\text{pr}}(H) \to H_{\mathcal{P}}$, $\sigma : C_{\text{pr}}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\text{pr}}(H) \to H_{\mathcal{P}\otimes\mathcal{E}}$. The initial condition is an $\mathcal{F}_0$-measurable random variable $h_0 : \Omega \to H$.

3.1. **Definition.** *A process $r \in C_{\text{pr}}(H)$ is called a* solution *for (3.1) with $r_0 = h_0$ if we have $\alpha(r) \in L^1_{\text{loc}}(\lambda; H)$, $\sigma(r) \in L^2_{\text{loc}}(W; L^0_2)$, $\gamma(r) \in L^2_{\text{loc}}(\mu; H)$ and*

$$\begin{aligned} r_t = h_0 &+ \int_0^t \alpha(r)_s ds + \int_0^t \sigma(r)_s dW_s \\ &+ \int_0^t \int_E \gamma(r)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0. \end{aligned}$$

By convention, uniqueness of solutions for (3.1) is meant up to indistinguishability, that is, for two solutions $r, \tilde{r}$ we have $\mathbb{P}(\bigcap_{t\in\mathbb{R}_+}\{r_t = \tilde{r}_t\}) = 1$.

3.2. **Remark.** *Note that in this defintion time-dependence of the vector fields is naturally included into the setting.*

The following two standard assumptions are crucial for existence and uniqueness:

3.3. **Assumption.** *Denoting by $\mathbf{0} \in C_{\text{pr}}(H)$ the zero process, we assume that*

$$\sup_{t\in[0,T]} \mathbb{E}[\|\alpha(\mathbf{0})_t\|^2] < \infty, \tag{3.2}$$

$$\sup_{t\in[0,T]} \mathbb{E}[\|\sigma(\mathbf{0})_t\|^2_{L^0_2}] < \infty, \tag{3.3}$$

$$\sup_{t\in[0,T]} \mathbb{E}\left[\int_E \|\gamma(\mathbf{0})(t,x)\|^2 F(dx)\right] < \infty \tag{3.4}$$

for all $T \in \mathbb{R}_+$.

3.4. **Assumption.** *We assume that there is a non-decreasing function* $L : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for all* $t \in \mathbb{R}_+$ *we have*

$$\|\alpha(r^1)_t - \alpha(r^2)_t\| \leq L(t)\|r^1_t - r^2_t\| \quad \mathbb{P}\text{–}a.s. \tag{3.5}$$

$$\|\sigma(r^1)_t - \sigma(r^2)_t\|_{L_2^0} \leq L(t)\|r^1_t - r^2_t\| \quad \mathbb{P}\text{–}a.s. \tag{3.6}$$

$$\bigg(\int_E \|\gamma(r^1)(t,x) - \gamma(r^2)(t,x)\|^2 F(dx) \bigg)^{\frac{1}{2}} \leq L(t)\|r^1_t - r^2_t\| \quad \mathbb{P}\text{–}a.s. \tag{3.7}$$

for all $r^1, r^2 \in C_{\mathrm{pr}}(H)$.

3.5. **Lemma.** *For each* $r \in C_{\mathrm{pr}}(H)$ *there is a non-decreasing function* $K_r : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for all* $T \in \mathbb{R}_+$ *and* $t_1, t_2 \in [0,T]$ *we have*

$$\mathbb{E}\bigg[\int_{t_1}^{t_2} \|\alpha(r)_s\|^2 ds \bigg] \leq K_r(T)|t_1 - t_2|, \tag{3.8}$$

$$\mathbb{E}\bigg[\int_{t_1}^{t_2} \|\sigma(r)_s\|^2_{L_2^0} ds \bigg] \leq K_r(T)|t_1 - t_2|, \tag{3.9}$$

$$\mathbb{E}\bigg[\int_{t_1}^{t_2} \int_E \|\gamma(r)(s,x)\|^2 F(dx) ds \bigg] \leq K_r(T)|t_1 - t_2|. \tag{3.10}$$

Proof. Let $r \in C_{\mathrm{pr}}(H)$ be arbitrary. Using the Lipschitz conditions (3.5), (3.6), (3.7) we obtain for all $T \in \mathbb{R}_+$ and $t_1, t_2 \in [0,T]$ the estimates

$$\mathbb{E}\bigg[\int_{t_1}^{t_2} \|\alpha(r)_s\|^2 ds \bigg] \leq 2|t_2 - t_1| L(T)^2 \|r\|^2_{T,0} + 2|t_2 - t_1| \sup_{t\in[0,T]} \mathbb{E}[\|\alpha(\mathbf{0})_t\|^2],$$

$$\mathbb{E}\bigg[\int_{t_1}^{t_2} \|\sigma(r)_s\|^2 ds \bigg] \leq 2|t_2 - t_1| L(T)^2 \|r\|^2_{T,0} + 2|t_2 - t_1| \sup_{t\in[0,T]} \mathbb{E}[\|\sigma(\mathbf{0})_t\|^2],$$

$$\begin{aligned} &\mathbb{E}\bigg[\int_{t_1}^{t_2} \int_E \|\gamma(r)(s,x)\|^2 F(dx) ds \bigg] \\ &\leq 2|t_2 - t_1| L(T)^2 \|r\|^2_{T,0} + 2|t_2 - t_1| \sup_{t\in[0,T]} \mathbb{E}\bigg[\int_E \|\gamma(\mathbf{0})(t,x)\|^2 F(dx) \bigg]. \end{aligned}$$

By virtue of (3.2), (3.3), (3.4) these estimates yield (3.8), (3.9), (3.10). □

In particular, for all $r \in C_{\mathrm{pr}}(H)$ we have $\alpha(r) \in L^2(\lambda; H)$, $\sigma(r) \in L^2(W; L_2^0)$ and $\gamma(r) \in L^2(\mu; H)$ by Lemma 3.5.

Let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ be arbitrary. For $r \in C_{\mathrm{pr}}(H)$ we set

$$\begin{aligned} \Lambda_{h_0}(r)_t &:= h_0 + \int_0^t \alpha(r)_s ds + \int_0^t \sigma(r)_s dW_s \\ &\quad + \int_0^t \int_E \gamma(r)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0. \end{aligned}$$

3.6. **Lemma.** Λ_{h_0} *maps* $C_{\mathrm{pr}}(H)$ *into itself, and for each* $r \in C_{\mathrm{pr}}(H)$ *and all* $T \in \mathbb{R}_+$ *we have*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|\Lambda_{h_0}(r)_t\|^2 \bigg] < \infty. \tag{3.11}$$

Proof. Let $r \in C_{\mathrm{pr}}(H)$ be arbitrary. By Hölder's inequality, the Itô-isometries (2.1), (2.4) and Lemma 3.5 we have $\Lambda_{h_0} \in C_{\mathrm{pr}}(H)$. Relation (3.11) is established by Hölder's inequality, Doob's martingale inequality [9, Thm. 3.8], the Itô-isometries (2.1), (2.4) and Lemma 3.5. □

3.7. **Lemma.** *For each $T \in \mathbb{R}_+$ and every $a > 3(T+2)L(T)^2$ the map Λ_{h_0} is a contraction on $C_{\mathrm{pr}}[0,T]$ with respect to the norm $\| \cdot \|_{T,a}$.*

Proof. Let $T \in \mathbb{R}_+$, $r^1, r^2 \in C_{\mathrm{pr}}[0,T]$ and $t \in [0,T]$ be arbitrary. By using Hölder's inequality and (3.5) we obtain

$$\mathbb{E}\left[\left\| \int_0^t (\alpha(r^1)_s - \alpha(r^2)_s) ds \right\|^2 \right] \leq t \mathbb{E}\left[\int_0^t \| \alpha(r^1)_s - \alpha(r^2)_s \|^2 ds \right]$$
$$= t \int_0^t \mathbb{E}[\| \alpha(r^1)_s - \alpha(r^2)_s \|^2] ds \leq t L(t)^2 \int_0^t \mathbb{E}[\| r_s^1 - r_s^2 \|^2] ds.$$

The Itô-isometry (2.1) and (3.6) yield

$$\mathbb{E}\left[\left\| \int_0^t (\sigma(r^1)_s - \sigma(r^2)_s) dW_s \right\|^2 \right] = \mathbb{E}\left[\int_0^t \| \sigma(r^1)_s - \sigma(r^2)_s \|_{L_2^0}^2 ds \right]$$
$$= \int_0^t \mathbb{E}[\| \sigma(r^1)_s - \sigma(r^2)_s \|_{L_2^0}^2] ds \leq L(t)^2 \int_0^t \mathbb{E}[\| r_s^1 - r_s^2 \|^2] ds,$$

and the Itô-isometry (2.4) and (3.7) give us

$$\mathbb{E}\left[\left\| \int_0^t \int_E (\gamma(r^1)(s,x) - \gamma(r^2)(s,x))(\mu(ds,dx) - F(dx)ds) \right\|^2 \right]$$
$$= \int_0^t \mathbb{E}\left[\int_E \| \gamma(r^1)(s,x) - \gamma(r^2)(s,x) \|^2 F(dx) \right] ds \leq L(t)^2 \int_0^t \mathbb{E}[\| r_s^1 - r_s^2 \|^2] ds.$$

Thus, we obtain for all $t \in [0,T]$ the estimate

$$\mathbb{E}[\| \Lambda_{h_0}(r^1)_t - \Lambda_{h_0}(r^2)_t \|^2] \leq 3(t+2)L(t)^2 \int_0^t \mathbb{E}[\| r_s^1 - r_s^2 \|^2] ds.$$

Now let $a > 3(T+2)L(T)^2$ be arbitrary. For each $t \in [0,T]$ we get

$$e^{-at} \mathbb{E}[\| \Lambda_{h_0}(r^1)_t - \Lambda_{h_0}(r^2)_t \|^2] \leq 3(T+2)L(T)^2 \int_0^t e^{-a(t-s)} e^{-as} \mathbb{E}[\| r_s^1 - r_s^2 \|^2] ds$$
$$\leq 3(T+2)L(T)^2 \sup_{s \in [0,t]} \left(e^{-as} \mathbb{E}[\| r_s^1 - r_s^2 \|^2] \right) \int_0^t e^{-a(t-s)} ds$$
$$\leq \frac{3(T+2)L(T)^2}{a} \sup_{t \in [0,T]} \left(e^{-at} \mathbb{E}[\| r_t^1 - r_t^2 \|^2] \right) = \frac{3(T+2)L(T)^2}{a} \| r^1 - r^2 \|_{T,a}^2.$$

This implies

$$\| \Lambda_{h_0}(r^1) - \Lambda_{h_0}(r^2) \|_{T,a} \leq \sqrt{\frac{3(T+2)L(T)^2}{a}} \| r^1 - r^2 \|_{T,a}.$$

Since $a > 3(T+2)L(T)^2$, we conclude that Λ_{h_0} is a contraction on $C_{\mathrm{pr}}[0,T]$ with respect to the norm $\| \cdot \|_{T,a}$. □

3.8. **Remark.** *The idea to change to an equivalent norm, in order to obtain a contraction, already appears in the proof* [14, Thm. 4.1] *which deals with infinite dimensional stochastic differential equations driven by Wiener processes.*

3.9. **Theorem.** *Suppose that Assumptions 3.3 and 3.4 are fulfilled. For each $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ there exists a unique càdlàg solution $r \in C_{\mathrm{pr}}(H)$ for (3.1) with $r_0 = h_0$ satisfying*

$$\mathbb{E}\left[\sup_{t \in [0,T]} \| r_t \|^2 \right] < \infty \quad \textit{for all } T \in \mathbb{R}_+. \tag{3.12}$$

Proof. Let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ be arbitrary. It suffices to prove existence and uniqueness on $[0,T]$ for each $T \in \mathbb{R}_+$. Let $T \in \mathbb{R}_+$ be arbitrary and choose $a > 3(T+2)L(T)^2$. Then, by Lemma 3.7, Λ_{h_0} is a contraction on $C_{\mathrm{pr}}[0,T]$ with respect to the norm $\| \cdot \|_{T,a}$. By the Banach fixed point theorem, there is a unique solution $r \in C_{\mathrm{pr}}[0,T]$ of the equation

$$r = \Lambda_{h_0}(r), \quad r \in C_{\mathrm{pr}}[0,T]. \tag{3.13}$$

This solution satisfies (3.12) by virtue of Lemma 3.6, and it has a càdlàg-version as the sum of stochastic integrals, see Section 2.4. □

4. L^p-ESTIMATES

For the SDEs of Section 3 the full theory of L^p estimates for solutions of stochastic differential equations holds true. We do not go into detail here but we simply want to point out several results, which are consequences of Burkholder-Davis-Gundy and Bichteler-Jacod type arguments (see [23, Lemma 3.1]).

Let $p \geq 2$ be arbitrary. Suppose that in addition to Assumptions 3.3, 3.4 we have

$$\sup_{t \in [0,T]} \mathbb{E}[\|\alpha(\mathbf{0})_t\|^p] < \infty, \tag{4.1}$$

$$\sup_{t \in [0,T]} \mathbb{E}[\|\sigma(\mathbf{0})_t\|_{L_2^0}^p] < \infty, \tag{4.2}$$

(4.3)

$$\sup_{t \in [0,T]} \mathbb{E}\left[\left(\int_E \|\gamma(\mathbf{0})(t,x)\|^2 F(dx) \right)^{\frac{p}{2}} \right] + \sup_{t \in [0,T]} \mathbb{E}\left[\int_E \|\gamma(\mathbf{0})(t,x)\|^p F(dx) \right] < \infty$$

for all $T \in \mathbb{R}_+$ as well as

$$\|\alpha(r)_t - \alpha(\mathbf{0})_t\|^p \leq L(t)\|r_t\|^2 \quad \mathbb{P}\text{–a.s.} \tag{4.4}$$

$$\|\sigma(r)_t - \sigma(\mathbf{0})_t\|_{L_2^0}^p \leq L(t)\|r_t\|^2 \quad \mathbb{P}\text{–a.s.} \tag{4.5}$$

$$\begin{aligned} &\left(\int_E \|\gamma(r)(t,x) - \gamma(\mathbf{0})(t,x)\|^2 F(dx) \right)^{\frac{p}{2}} + \\ &+ \int_E \|\gamma(r)(t,x) - \gamma(\mathbf{0})(t,x)\|^p F(dx) \leq L(t)\|r_t\|^2 \quad \mathbb{P}\text{–a.s.} \end{aligned} \tag{4.6}$$

for all $t \in \mathbb{R}_+$ and all $r \in C_{\mathrm{pr}}(H)$.

Then, for each $h_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ the solution r for (3.1) with $r_0 = h_0$ satisfies the L^p-estimate

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \|r_t\|^p \Big] < \infty \quad \text{for all } T \in \mathbb{R}_+. \tag{4.7}$$

Indeed, let $T \in \mathbb{R}_+$ be arbitrary. By Hölder's inequality we obtain

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [0,T]} \left\| \int_0^t \alpha(r)_s ds \right\|^p \right] &\leq \mathbb{E}\left[\left(\int_0^T \|\alpha(r)_s\| ds \right)^p \right] \\ &\leq T^{p-1} 2^{p-1} \mathbb{E}\Big[\int_0^T (\|\alpha(r)_s - \alpha(\mathbf{0})_s\|^p + \|\alpha(\mathbf{0})_s\|^p) ds \Big]. \end{aligned}$$

By the Burkholder-Davis-Gundy inequality and Hölder's inequality we have

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [0,T]} \left\| \int_0^t \sigma(r)_s dW_s \right\|^p \right] &\leq C_p \mathbb{E}\left[\left(\int_0^T \|\sigma(r)_s\|_{L_2^0}^2 ds \right)^{\frac{p}{2}} \right] \\ &\leq C_p T^{\frac{p}{2}-1} 2^{p-1} \mathbb{E}\Big[\int_0^T (\|\sigma(r)_s - \sigma(\mathbf{0})_s\|_{L_2^0}^p + \|\sigma(\mathbf{0})_s\|_{L_2^0}^p) ds \Big] \end{aligned}$$

with a constant $C_p > 0$ as well as by means of the Jacod-Bichteler inequality (see [23, Lemma 2.3])

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t\int_E \gamma(r)(s,x)(\mu(ds,dx)-F(dx)ds)\right\|^p\right] \\
&\leq N\int_0^T \mathbb{E}\left[\left(\int_E \|\gamma(r)(s,x)\|^2F(dx)\right)^{\frac{p}{2}} + \int_E \|\gamma(r)(s,x)\|^pF(dx)\right]ds \\
&\leq N2^{p-1}\int_0^T \mathbb{E}\left[\left(\int_E \|\gamma(r)(s,x)-\gamma(\mathbf{0})(s,x)\|^2F(dx)\right)^{\frac{p}{2}}\right]ds+ \\
&\quad + N2^{p-1}\int_0^T \mathbb{E}\left[\int_E \|\gamma(r)(s,x)-\gamma(\mathbf{0})(s,x)\|^pF(dx)\right]ds+ \\
&\quad + N2^{p-1}\int_0^T \mathbb{E}\left[\left(\int_E \|\gamma(\mathbf{0})(s,x)\|^2F(dx)\right)^{\frac{p}{2}}\right]ds+ \\
&\quad + N2^{p-1}\int_0^T \mathbb{E}\left[\int_E \|\gamma(\mathbf{0})(s,x)\|^pF(dx)\right]ds.
\end{aligned}
$$

These three estimates, together with (4.1)–(4.6), prove that (4.7) is valid.

4.1. **Remark.** *Alternatively one could apply for the stochastic integral with respect to the Poisson measure Burkholder-Davis-Gundy type estimates, however, then one has to change Assumption 4.1 and 4.6 by the respective assertions on the quadratic variation, namely*

$$\sup_{t\in[0,T]}\mathbb{E}\left[\left(\int_0^t\int_E \|\gamma(\mathbf{0})(t,x)\|^2\mu(dt,dx)\right)^{\frac{p}{2}}\right] < \infty,$$

$$\mathbb{E}\left[\left(\int_0^t\int_E \|\gamma(r)(t,x)-\gamma(\mathbf{0})(t,x)\|^2\mu(ds,dx)\right)^{\frac{p}{2}}\right] \leq L(t)\mathbb{E}\left[\int_0^t \|r_t\|^2dt\right] \quad \mathbb{P}\text{–}a.s.$$

for all $t \in \mathbb{R}_+$ and all $r \in C_{\mathrm{pr}}(H)$.

4.2. **Remark.** *One can generalize the whole theory to the L^p-setting by replacing Assumptions 3.3 and 3.4 by the corresponding L^p-versions of it and applying the previous reasonings directly (following for instance* [23]*). This would give the general L^p-theory for strong solutions of SDEs.*

5. Stability of stochastic differential equations

We shall now deal with stability of stochastic differential equations of the kind (3.1). Again these are standard results which we do only give for the sake of completeness. Note that for each $B \in \mathcal{E}$ the random measure μ_B defined as

$$\mu_B(A) := \mu(A \cap \mathbb{R}_+ \times B), \quad A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$$

is again a Poisson random measure on $\mathbb{R}_+ \times E$. Its compensator is given by $dt \otimes F_B(dx)$ with F_B being defined as $F_B(A) := F(A \cap B)$, $A \in \mathcal{E}$.

As in Section 3, we assume that $\alpha : C_{\mathrm{pr}}(H) \to H_{\mathcal{P}}$, $\sigma : C_{\mathrm{pr}}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\mathrm{pr}}(H) \to H_{\mathcal{P}\otimes\mathcal{E}}$ fulfill Assumptions 3.3 and 3.4. Furthermore, let, for each $n \in \mathbb{N}$, $\alpha_n : C_{\mathrm{pr}}(H) \to H_{\mathcal{P}}$, $\sigma_n : C_{\mathrm{pr}}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma_n : C_{\mathrm{pr}}(H) \to H_{\mathcal{P}\otimes\mathcal{E}}$ be given. We make the following additional assumptions.

5.1. **Assumption.** *Denoting by $\mathbf{0} \in C_{\mathrm{pr}}(H)$ the zero process, we assume that*

$$\sup_{t \in [0,T]} \mathbb{E}[\|\alpha_n(\mathbf{0})_t\|^2] < \infty,$$
$$\sup_{t \in [0,T]} \mathbb{E}[\|\sigma_n(\mathbf{0})_t\|_{L_2^0}^2] < \infty,$$
$$\sup_{t \in [0,T]} \mathbb{E}\bigg[\int_E \|\gamma_n(\mathbf{0})(t,x)\|^2 F(dx) \bigg] < \infty$$

for each $n \in \mathbb{N}$ and for all $T \in \mathbb{R}_+$.

5.2. **Assumption.** *We assume that for all $t \in \mathbb{R}_+$ we have*

$$\|\alpha_n(r^1)_t - \alpha_n(r^2)_t\| \leq L(t)\|r_t^1 - r_t^2\| \quad \mathbb{P}\text{–}a.s. \tag{5.1}$$

$$\|\sigma_n(r^1)_t - \sigma_n(r^2)_t\|_{L_2^0} \leq L(t)\|r_t^1 - r_t^2\| \quad \mathbb{P}\text{–}a.s. \tag{5.2}$$

$$\bigg(\int_E \|\gamma_n(r^1)(t,x) - \gamma_n(r^2)(t,x)\|^2 F(dx) \bigg)^{\frac{1}{2}} \leq L(t)\|r_t^1 - r_t^2\| \quad \mathbb{P}\text{–}a.s. \tag{5.3}$$

for all $r^1, r^2 \in C_{\mathrm{pr}}(H)$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the non-decreasing function from Assumption 3.4.

5.3. **Remark.** *Notice the slight difference of the previous assumptions to Assumptions 3.3 and 3.4 for each α_n, σ_n and γ_n, namely, that the constant L does not depend on $n \geq 1$.*

Furthermore, let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and for each $n \in \mathbb{N}$ let $h_0^n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $B_n \in \mathcal{E}$ be given.

According to Theorem 3.9, there exists a unique càdlàg solution $r \in C_{\mathrm{pr}}(H)$ for (3.1) with $r_0 = h_0$ satisfying (3.12), and for each $n \in \mathbb{N}$ there exists a unique càdlàg solution $r^n \in C_{\mathrm{pr}}(H)$ for

$$\begin{cases} dr_t^n &= \alpha_n(r^n)_t dt + \sigma_n(r^n)_t dW_t + \int_E \gamma_n(r^n)(t,x)(\mu_{B_n}(dt,dx) - F_{B_n}(dx)dt) \\ r_0^n &= h_0^n, \end{cases}$$

satisfying $\mathbb{E}[\sup_{t \in [0,T]} \|r_t^n\|^2] < \infty$ for all $T \in \mathbb{R}_+$.

We also make the following assumption, in which $r \in C_{\mathrm{pr}}(H)$ denotes the solution for (3.1) with $r_0 = h_0$.

5.4. **Assumption.** *We assume that $h_0^n \to h_0$ in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, $\alpha_n(r) \to \alpha(r)$ in $L^2(\lambda; H)$, $\sigma_n(r) \to \sigma(r)$ in $L^2(W; L_2^0)$, $\gamma_n(r) \to \gamma(r)$ in $L^2(\mu; H)$ and $B_n \uparrow E$.*

Notice that, by Assumption 5.4, we have

$$C_n(t,r) := \mathbb{E}[\|h_0 - h_0^n\|^2] + tA_n(t,r) + 4\Sigma_n(t,r) + 4\Gamma_n(t,r) + 4G_n(t,r) \to 0$$

as $n \to \infty$ for all $t \in \mathbb{R}_+$, where we have set

$$A_n(t,r) := \mathbb{E}\bigg[\int_0^t \|\alpha(r)_s - \alpha_n(r)_s\|^2 ds \bigg], \tag{5.4}$$

$$\Sigma_n(t,r) := \mathbb{E}\bigg[\int_0^t \|\sigma(r)_s - \sigma_n(r)_s\|_{L_2^0}^2 ds \bigg], \tag{5.5}$$

$$\Gamma_n(t,r) := \mathbb{E}\bigg[\int_0^t \int_E \|\gamma(r)(s,x) - \gamma_n(r)(s,x)\|^2 F(dx) ds \bigg], \tag{5.6}$$

$$G_n(t,r) := \mathbb{E}\bigg[\int_0^t \int_{E \setminus B_n} \|\gamma(r)(s,x)\|^2 F(dx) ds \bigg]. \tag{5.7}$$

5.5. **Proposition.** *Suppose that Assumptions 5.1, 5.2 and 5.4 are fulfilled. Then for each $t \in \mathbb{R}_+$ the estimate*

$$\mathbb{E}[\|r_t - r_t^n\|^2] \leq 8e^{8t(t+8)L(t)^2} C_n(t,r) \to 0 \quad \text{for } n \to \infty \tag{5.8}$$

is valid, and for every $T \in \mathbb{R}_+$ we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|r_t - r_t^n\|^2\right] \leq 8\left(1 + 8T(T+8)L(T)^2 e^{8T(T+8)L(T)^2}\right) C_n(T,r) \to 0 \tag{5.9}$$

for $n \to \infty$.

Proof. Let $T \in \mathbb{R}_+$ and $n \in \mathbb{N}$ be arbitrary. By Hölder's inequality, Doob's martingale inequality [9, Thm. 3.8], the Itô-isometries (2.1), (2.4) and the Lipschitz conditions (5.1), (5.2), (5.3) we obtain for all $t \in [0,T]$ the estimate

(5.10)

$$\begin{aligned}
&\mathbb{E}[\|r_t - r_t^n\|^2] \leq \mathbb{E}\left[\sup_{s\in[0,t]} \|r_s - r_s^n\|^2\right] \leq 8C_n(t,r) \\
&\quad + 8\mathbb{E}\left[\sup_{s\in[0,t]} \left\| \int_0^t (\alpha_n(r)_s - \alpha_n(r^n)_s) ds \right\|^2\right] \\
&\quad + 8\mathbb{E}\left[\sup_{s\in[0,t]} \left\| \int_0^t (\sigma_n(r)_s - \sigma_n(r^n)_s) dW_s \right\|^2\right] \\
&\quad + 8\mathbb{E}\left[\sup_{s\in[0,t]} \left\| \int_0^t (\gamma_n(r)(s,x) - \gamma_n(r^n)(s,x))(\mu_{B_n}(ds,dx) - F_{B_n}(dx)ds) \right\|^2\right] \\
&\leq 8C_n(t,r) + 8(t+8)L(t)^2 \int_0^t \mathbb{E}[\|r_s - r_s^n\|^2] ds.
\end{aligned}$$

Applying the Gronwall Lemma gives us (5.8). In particular

$$\sup_{t\in[0,T]} \mathbb{E}[\|r_t - r_t^n\|^2] \leq 8e^{8T(T+8)L(T)^2} C_n(T,r), \quad n \in \mathbb{N}. \tag{5.11}$$

Combining (5.10) and (5.11) we obtain (5.9). □

6. Regular dependence on initial data for stochastic differential equations

We understand the question of regular dependence on initial data as a conclusion of the stability results of Section 5. We consider

$$\begin{cases} dr_t &= \alpha(r)_t dt + \sigma(r)_t dW_t + \int_E \gamma(r)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &= h_0, \end{cases} \tag{6.1}$$

under Assumptions 3.3 and 3.4, such that we can conclude the existence and uniqueness of strong solution for $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. Motivated by ideas from convenient analysis, see [21], we fix a curve of initial data $\epsilon \mapsto c(\epsilon) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, which is differentiable for all ϵ with derivative $c'(\epsilon) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. We consider the following system of equations,

(6.2)

$$\begin{cases} dr_t^\epsilon &= \alpha(r^\epsilon)_t dt + \sigma(r^\epsilon)_t dW_t + \int_E \gamma(r^\epsilon)(t,x)(\mu(dt,dx) - F(dx)dt), \\ r_0^\epsilon &= c(\epsilon), \\ d\frac{r_t^\epsilon - r_t^0}{\epsilon} &= \frac{\alpha(r^\epsilon)_t - \alpha(r^0)_t}{\epsilon} dt + \frac{\sigma(r^\epsilon)_t - \sigma(r^0)_t}{\epsilon} dW_t + \\ &\quad + \int_E \frac{\gamma(r^\epsilon)(t,x) - \gamma(r^0)(t,x)}{\epsilon}(\mu(dt,dx) - F(dx)dt), \\ \frac{r_0^\epsilon - r_0^0}{\epsilon} &= \frac{c(\epsilon) - c(0)}{\epsilon}, \end{cases}$$

for $\epsilon \neq 0$. We can consider those equations indeed as two SDEs in our sense. More precisely let

$$\Delta_t^\epsilon := \frac{r_t^\epsilon - r_t^0}{\epsilon} \tag{6.3}$$

and

$$\left\{ \begin{array}{rcl} dr_t^\epsilon & = & \alpha(r^\epsilon)_t dt + \sigma(r^\epsilon)_t dW_t + \int_E \gamma(r^\epsilon)(t,x)(\mu(dt,dx) - F(dx)dt), \\ r_0^\epsilon & = & c(\epsilon), \\ d\Delta_t^\epsilon & = & \frac{\alpha(\epsilon\Delta^\epsilon + r^0)_t - \alpha(r^0)_t}{\epsilon} dt + \frac{\sigma(\epsilon\Delta^\epsilon + r^0)_t - \sigma(r^0)_t}{\epsilon} dW_t + \\ & & + \int_E \frac{\gamma(\epsilon\Delta^\epsilon + r^0)(t,x) - \gamma(r^0)(t,x)}{\epsilon}(\mu(dt,dx) - F(dx)dt), \\ \Delta_0^\epsilon & = & \frac{c(\epsilon) - c(0)}{\epsilon}, \end{array} \right. \tag{6.4}$$

for $\epsilon \neq 0$, then this system of equations can be seen as two stochastic differential equations. We can readily check that the Assumptions 3.3 and 3.4 are true for the second SDE for every $\epsilon \neq 0$.

We assume now that the maps α, σ and γ admit directional derivatives in all directions of $C_{\mathrm{pr}}(H)$, where those maps are defined. We denote those directional derivatives at the point $r \in C_{\mathrm{pr}}(H)$ into direction $v \in C_{\mathrm{pr}}(H)$ by $D\alpha(r)\bullet v$, $D\sigma(r)\bullet v$ and $D\gamma(r,.,.) \bullet v$.

6.1. **Assumption.** *We define the first variation process $J(r) \bullet w$ in direction w, where $w \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, to satisfy the SDE*

(6.5)

$$\left\{ \begin{array}{rcl} d(J(r) \bullet w)_t & = & \big(D\alpha(r) \bullet (J(r) \bullet w)\big)_t dt + \big(D\sigma(r) \bullet (J(r) \bullet w)\big)_t dW_t + \\ & & + \int_E \big(D\gamma(r) \bullet (J(r) \bullet w)\big)(t,x)(\mu(dt,dx) - F(dx)dt), \\ (J(r) \bullet w)_0 & = & w, \end{array} \right.$$

where r solves equation (6.1)*. We assume that the Assumptions 3.3 and 3.4 are true for this equation. We assume furthermore that*

$$\frac{\alpha(\epsilon(J(r) \bullet w) + r) - \alpha(r)}{\epsilon} \to D\alpha(r) \bullet (J(r) \bullet w),$$
$$\frac{\sigma(\epsilon(J(r) \bullet w) + r) - \sigma(r)}{\epsilon} \to D\sigma(r) \bullet (J(r) \bullet w),$$
$$\frac{\gamma(\epsilon(J(r) \bullet w) + r) - \gamma(r)}{\epsilon} \to D\gamma(r) \bullet (J(r) \bullet w)$$

as $\epsilon \to 0$ in the respective spaces. The process r denotes the solution of equation 6.1 and $J(r) \bullet w$ denotes the solution of the first variation equation (6.5)*.*

6.2. **Proposition.** *Suppose that Assumptions 5.1, 5.2 for equation*

$$\left\{ \begin{array}{rcl} d\Delta_t^\epsilon & = & \frac{\alpha(\epsilon\Delta^\epsilon + r^0)_t - \alpha(r^0)_t}{\epsilon} dt + \frac{\sigma(\epsilon\Delta^\epsilon + r^0)_t - \sigma(r^0)_t}{\epsilon} dW_t + \\ & & + \int_E \frac{\gamma(\epsilon\Delta^\epsilon + r^0)(t,x) - \gamma(r^0)(t,x)}{\epsilon}(\mu(dt,dx) - F(dx)dt), \\ \Delta_0^\epsilon & = & \frac{c(\epsilon) - c(0)}{\epsilon}, \end{array} \right. \tag{6.6}$$

are valid in the obvious sense for $\epsilon \neq 0$ in a neighborhood of 0, and assume that 6.1 is fulfilled for $w = c'(0)$ for a chosen curve of initial values $\epsilon \mapsto c(\epsilon)$. Then for each $t \in \mathbb{R}_+$ the estimate

$$\mathbb{E}[\|(J(r) \bullet w)_t - \Delta_t^\epsilon\|^2] \leq 8e^{8t(t+8)L(t)^2} C_\epsilon(t,r) \to 0 \quad \textit{for } \epsilon \to 0 \tag{6.7}$$

is valid, and for every $T \in \mathbb{R}_+$ we have

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \| (J(r) \bullet w)_t - \Delta_t^{\epsilon} \|^2 \Big] \leq 8 \left(1 + 8T(T+8)L(T)^2 e^{8T(T+8)L(T)^2} \right) C_{\epsilon}(T,r) \to 0 \tag{6.8}$$

for $\epsilon \to 0$. In particular the map $w \mapsto J(r) \bullet w$ is linear and continuously depending on w in the sense that for every $T \in \mathbb{R}_+$ we have

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \| (J(r) \bullet w_n)_t - (J(r) \bullet w)_t \|^2 \Big] \leq 8 \left(1 + 8T(T+8)L(T)^2 e^{8T(T+8)L(T)^2} \right) \mathbb{E}[\|w - w^n\|^2] \to 0 \tag{6.9}$$

for variation of the initial value $w^n \to w \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$.

6.3. **Remark.** *The notions L and C_ϵ are defined corresponding to Assumptions 5.1 and 5.2.*

Proof. The assertion is a corollary of Proposition 5.5. Assumption 6.1 corresponds precisely to Assumption 5.4, which is needed for the proof of Proposition 5.5. Note that the "continuous" parameter ϵ replaces the index n, which does not cause any problems, since we do not speak about almost sure convergence results here. □

Considering the construction for all possible curves of initial values c we can define the first (and possibly higher) variation processes in a coherent way for all variations of the initial values and also for variations of the process up to time t by shifting $\mathcal{F}_t$ to $\mathcal{F}_0$. Properties of this variation process can be established by considering the equation, which follows right from Proposition 6.2,

$$r^{\epsilon} - r^0 = \int_0^{\epsilon} J(r) \bullet c'(\epsilon)\, d\epsilon \tag{6.10}$$

and which reveals the true meaning of the first variation process.

7. Solution concepts for Stochastic partial differential equations

When dealing with SPDEs there are several solution concepts, which will be discussed in this section. The main difficulty is that solutions of SPDEs usually leave the realm of semi-martingales and one therefore has to modify the usual semi-martingale decomposition. The method of the moving frame, which will be presented in the next section, is a new approach how to handle this problem.

Now let $(S_t)_{t \geq 0}$ be a C_0-semigroup on the separable Hilbert space H with infinitesimal generator $A : \mathcal{D}(A) \subset H \to H$. We denote by $A^* : \mathcal{D}(A^*) \subset H \to H$ the adjoint operator of A. Recall that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense in H, see, e.g., [31, Satz VII.4.6, p. 351].

In this section, we are interested in stochastic partial differential equations of the form

$$\begin{cases} dr_t &= (Ar_t + \alpha(r)_t)dt + \sigma(r)_t dW_t + \int_E \gamma(r)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &= h_0 \end{cases} \tag{7.1}$$

where $\alpha : C_{\rm pr}(H) \to H_{\mathcal{P}}$, $\sigma : C_{\rm pr}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\rm pr}(H) \to H_{\mathcal{P} \otimes \mathcal{E}}$. The initial condition is an $\mathcal{F}_0$-measurable random variable $h_0 : \Omega \to H$.

For each j we define $\sigma^j : C_{\rm pr}(H) \to H_{\mathcal{P}}$ as $\sigma^j(r)_t := \sqrt{\lambda_j} \sigma(r)_t e_j$.

7.1. **Definition.** *A process* $r \in C_{\mathrm{pr}}(H)$ *is called a* strong solution *for (7.1) with* $r_0 = h_0$ *if we have* $r_t \in \mathcal{D}(A)$, $t \geq 0$, *the relations* $Ar + \alpha(r) \in L^1_{\mathrm{loc}}(\lambda; H)$, $\sigma(r) \in L^2_{\mathrm{loc}}(W; L^2_0)$, $\gamma(r) \in L^2_{\mathrm{loc}}(\mu; H)$ *and*

$$
\begin{aligned}
r_t = h_0 + \int_0^t (Ar_s + \alpha(r)_s) ds + \sum_j \int_0^t \sigma^j(r)_s d\beta_s^j \\
+ \int_0^t \int_E \gamma(r)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0.
\end{aligned} \tag{7.2}
$$

7.2. **Definition.** *A process* $r \in C_{\mathrm{pr}}(H)$ *is called a* weak solution *for (7.1) with* $r_0 = h_0$ *if* $\alpha(r) \in L^1_{\mathrm{loc}}(\lambda; H)$, $\sigma(r) \in L^2_{\mathrm{loc}}(W; L^2_0)$, $\gamma(r) \in L^2_{\mathrm{loc}}(\mu; H)$ *and for all* $\xi \in \mathcal{D}(A^*)$ *we have*

$$
\begin{aligned}
\langle \xi, r_t \rangle = \langle \xi, h_0 \rangle + \int_0^t (\langle A^* \xi, r_s \rangle + \langle \xi, \alpha(r)_s \rangle) ds + \sum_j \int_0^t \langle \xi, \sigma^j(r)_s \rangle d\beta_s^j \\
+ \int_0^t \int_E \langle \xi, \gamma(r)(s,x) \rangle (\mu(ds,dx) - F(dx)ds), \quad t \geq 0.
\end{aligned} \tag{7.3}
$$

7.3. **Definition.** *A process* $r \in C_{\mathrm{pr}}(H)$ *is called a* mild solution *for (7.1) with* $r_0 = h_0$ *if* $\alpha(r) \in L^1_{\mathrm{loc}}(\lambda; H)$, $\sigma(r) \in L^2_{\mathrm{loc}}(W; L^2_0)$, $\gamma(r) \in L^2_{\mathrm{loc}}(\mu; H)$ *and we have*

$$
\begin{aligned}
r_t = S_t h_0 + \int_0^t S_{t-s} \alpha(r)_s ds + \sum_j \int_0^t S_{t-s} \sigma^j(r)_s d\beta_s^j \\
+ \int_0^t \int_E S_{t-s} \gamma(r)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0.
\end{aligned} \tag{7.4}
$$

Again, *uniqueness* of solutions for (7.1) is meant up to indistinguishability, that is, for two solutions $r, \tilde{r}$ we have $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{r_t = \tilde{r}_t\}) = 1$.

We can express equations (7.2), (7.3) and (7.4) by the Q-Wiener process W as follows.

(1) According to (2.2) we have

$$
\int_0^t \sigma(r)_s dW_s = \sum_j \int_0^t \sigma^j(r)_s d\beta_s^j, \quad t \in \mathbb{R}_+ \tag{7.5}
$$

(2) For each $\xi \in \mathcal{D}(A^*)$ the U_0'-valued process $\tilde{\sigma}$ defined as

$$
\tilde{\sigma}(r)_s(u) := \langle \xi, \sigma(r)_s \rangle (u) := \langle \xi, \sigma(r)_s(u) \rangle, \quad u \in U_0
$$

has values in $L_2(U_0; \mathbb{R})$. For every j we set $\tilde{\sigma}^j(r)_s := \sqrt{\lambda_j} \tilde{\sigma}(r)_s e_j$. Noting that

$$
\tilde{\sigma}^j(r)_s = \langle \xi, \sigma^j(r)_s \rangle, \quad s \in \mathbb{R}_+
$$

we deduce, by using (2.2), the identity

$$
\int_0^t \langle \xi, \sigma(r)_s \rangle dW_s = \sum_j \int_0^t \langle \xi, \sigma^j(r)_s \rangle d\beta_s^j, \quad t \in \mathbb{R}_+. \tag{7.6}
$$

(3) For fixed $t \in \mathbb{R}_+$, the $L(U_0; H)$-valued process

$$
\hat{\sigma}(r)_s := S_{t-s} \sigma(r)_s, \quad s \in [0,t]
$$

has values in L_2^0. For every j we set $\hat{\sigma}^j(r)_s := \sqrt{\lambda_j} \hat{\sigma}(r)_s e_j$. Noting that

$$
\hat{\sigma}^j(r)_s = S_{t-s} \sigma^j(r)_s, \quad s \in [0,t]
$$

we deduce, by using (2.2), the identity

$$\int_0^t S_{t-s}\sigma(r)_s dW_s = \sum_j \int_0^t S_{t-s}\sigma^j(r)_s d\beta_s^j, \quad t \in \mathbb{R}_+. \tag{7.7}$$

In (7.5), (7.6) and (7.7) the convergence is uniformly on compact time intervals in probability.

7.4. **Lemma.** *Let $r \in C_{\mathrm{pr}}(H)$ be a strong solution for (7.1) with $r_0 = h_0$. Then, r is also a weak solution for (7.1) with $r_0 = h_0$.*

Proof. For all $\xi \in \mathcal{D}(A^*)$ we have

$$\begin{aligned}\langle \xi, r_t \rangle &= \langle \xi, h_0 \rangle + \int_0^t \langle \xi, Ar_s + \alpha(r)_s \rangle ds + \sum_j \int_0^t \langle \xi, \sigma^j(r)_s \rangle d\beta_s^j \\ &\quad + \int_0^t \int_E \langle \xi, \gamma(r)(s,x) \rangle (\mu(ds,dx) - F(dx)ds), \quad t \geq 0\end{aligned} \tag{7.8}$$

implying that r is also a weak solution for (7.1) with $r_0 = h_0$, because $\langle \xi, Ah \rangle = \langle A^*\xi, h \rangle$ for all $h \in \mathcal{D}(A)$. □

7.5. **Lemma.** *Let $r \in C_{\mathrm{pr}}(H)$ be a weak solution for (7.1) with $r_0 = h_0$. Then, r is also a mild solution for (7.1) with $r_0 = h_0$.*

Proof. Let $\xi \in \mathcal{D}(A^*)$ be arbitrary. We fix an arbitrary $T \in \mathbb{R}_+$ and $g \in C^1([0,T];\mathbb{R})$. By the definition of the quadratic co-variation, see e.g. [18, Def. I.4.45], we obtain

$$\langle g(t)\xi, r_t \rangle = \langle g(0)\xi, h_0 \rangle + \int_0^t g(s) d\langle \xi, r_s \rangle + \int_0^t \langle \xi, r_s \rangle dg(s) + [g, \langle \xi, r \rangle]_t.$$

Since $g \in C^1([0,T];\mathbb{R})$, we have $[g, \langle \xi, r \rangle] = 0$ according to [18, Prop. 4.49.d]. Therefore and because of (7.3), we get

$$\begin{aligned}\langle g(t)\xi, r_t \rangle &= \langle g(0)\xi, h_0 \rangle + \int_0^t \Big(\langle g'(s)\xi + A^* g(s)\xi, r_s \rangle + \langle g(s)\xi, \alpha(r)_s \rangle \Big) ds \\ &\quad + \sum_j \int_0^t \langle g(s)\xi, \sigma^j(r)_s \rangle d\beta_s^j + \int_0^t \int_E \langle g(s)\xi, \gamma(r)(s,x) \rangle (\mu(ds,dx) - F(dx)ds).\end{aligned}$$

Since the set $\{t \mapsto g(t)\xi \,|\, g \in C^1([0,T];\mathbb{R}) \text{ and } \xi \in \mathcal{D}(A^*)\}$ is linearly dense in $C^1([0,T];\mathcal{D}(A^*))$, we deduce

$$\begin{aligned}\langle g(t), r_t \rangle &= \langle g(0), h_0 \rangle + \int_0^t \Big(\langle g'(s) + A^* g(s), r_s \rangle + \langle g(s), \alpha(r)_s \rangle \Big) ds \\ &\quad + \sum_j \int_0^t \langle g(s), \sigma^j(r)_s \rangle d\beta_s^j + \int_0^t \int_E \langle g(s), \gamma(r)(s,x) \rangle (\mu(ds,dx) - F(dx)ds)\end{aligned}$$

for all $g \in C^1([0,T];\mathcal{D}(A^*))$, where we recall that $T \in \mathbb{R}_+$ was arbitrary. Defining $g \in C^1([0,t];\mathcal{D}(A^*))$ for an arbitrary $t \in \mathbb{R}_+$ and an arbitrary $\xi \in \mathcal{D}(A^*)$ as $g(s) := S_{t-s}^*\xi$, $s \in [0,t]$, we obtain $g'(s) = -A^* g(s)$, and hence

$$\begin{aligned}\langle \xi, r_t \rangle &= \langle \xi, S_t h_0 \rangle + \int_0^t \langle \xi, S_{t-s}\alpha(r)_s \rangle ds + \sum_j \int_0^t \langle \xi, S_{t-s}\sigma^j(r)_s \rangle d\beta_s^j \\ &\quad + \int_0^t \int_E \langle \xi, S_{t-s}\gamma(r)(s,x) \rangle (\mu(ds,dx) - F(dx)ds).\end{aligned}$$

Since $\mathcal{D}(A^*)$ is dense in H, the process r is also a mild solution to (7.1). □

For the proof of the upcoming result, we introduce the spaces

$$L^1_T(\lambda^2) := L^1([0,T]\times[0,T], \mathcal{B}[0,T]\otimes\mathcal{B}[0,T], \lambda\otimes\lambda), \tag{7.9}$$

$$L^2_T(\mathbb{P}\otimes\lambda^2) := L^2(\Omega\times[0,T]\times[0,T], \mathcal{P}_T\otimes\mathcal{B}[0,T], \mathbb{P}\otimes\lambda\otimes\lambda) \tag{7.10}$$

for each $T\in\mathbb{R}_+$.

7.6. **Lemma.** *Let $r\in C_{\mathrm{pr}}(H)$ be a mild solution for (7.1) with $r_0=h_0$ such that $\sigma(r)\in L^2(W;L^0_2)$ and $\gamma(r)\in L^2(\mu;H)$. Then, r is also a weak solution for (7.1) with $r_0=h_0$.*

Proof. Let $\xi\in\mathcal{D}(A^*)$ be arbitrary. Introducing the processes

$$\Pi_t := \int_0^t S_{t-s}\alpha(r)_s ds, \quad \Phi^j_t := \int_0^t S_{t-s}\sigma^j(r)_s d\beta^j_s$$
$$\text{and}\quad \Psi_t := \int_0^t\int_E S_{t-s}\gamma(r)(s,x)(\mu(ds,dx)-F(dx)ds),$$

we have, since r is a mild solution for (7.1) with $r_0=h_0$,

$$\langle\xi,r_t\rangle = \langle\xi,S_t h_0\rangle + \langle\xi,\Pi_t\rangle + \Big\langle\xi,\sum_j\Phi^j_t\Big\rangle + \langle\xi,\Psi_t\rangle,\quad t\geq 0. \tag{7.11}$$

Let $T\in\mathbb{R}_+$ be arbitrary. We introduce the process $\pi:\Omega\times[0,T]\times[0,T]\to\mathbb{R}$ as

$$\pi(t,s) := \begin{cases}\langle A^*\xi, S_{s-t}\alpha(r)_t\rangle, & s\geq t\\ 0, & s<t,\end{cases}$$

for each j the process $\phi^j:\Omega\times[0,T]\times[0,T]\to\mathbb{R}$ as

$$\phi^j(t,s) := \begin{cases}\langle A^*\xi, S_{s-t}\sigma^j(r)_t\rangle, & s\geq t\\ 0, & s<t,\end{cases}$$

and the process $\psi:\Omega\times[0,T]\times E\times[0,T]\to\mathbb{R}$ as

$$\psi(t,x,s) := \begin{cases}\langle A^*\xi, S_{s-t}\gamma(r)(t,x)\rangle, & s\geq t\\ 0, & s<t.\end{cases}$$

From the hypothesis we obtain $\pi\in L^1_T(\lambda^2)$ almost surely, $\phi\in L^2_T(\mathbb{P}\otimes\lambda^2)$ and $\psi\in L^2_T(\mu\otimes\lambda)$, see definitions (7.9), (7.10) and (A.2), which allows us to apply the stochastic Fubini theorems in the sequel.

Using Theorem A.2 and [31, Lemma VII.4.5(a)] we obtain

$$\begin{aligned}
\int_0^T\langle A^*\xi,\Psi_s\rangle ds &= \int_0^T\bigg(\int_0^T\int_E\psi(t,x,s)(\mu(dt,dx)-F(dx)dt)\bigg)ds\\
&= \int_0^T\int_E\bigg(\int_0^T\psi(t,x,s)ds\bigg)(\mu(dt,dx)-F(dx)dt)\\
&= \int_0^T\int_E\Big\langle A^*\xi,\int_t^T S_{s-t}\gamma(r)(t,x)ds\Big\rangle(\mu(dt,dx)-F(dx)dt)\\
&= \int_0^T\int_E\Big\langle \xi, A\int_0^{T-t} S_s\gamma(r)(t,x)ds\Big\rangle(\mu(dt,dx)-F(dx)dt)\\
&= \int_0^T\int_E\langle\xi,S_{T-t}\gamma(r)(t,x)-\gamma(r)(t,x)\rangle(\mu(dt,dx)-F(dx)dt)\\
&= \langle\xi,\Psi_T\rangle - \int_0^T\int_E\langle\xi,\gamma(r)(t,x)\rangle(\mu(dt,dx)-F(dx)dt).
\end{aligned}\tag{7.12}$$

An analogous calculation, using the stochastic Fubini theorem with respect to semimartingales, see, e.g., [25, Thm. IV.65], gives us, for each j, the identity

$$\int_0^T \langle A^* \xi, \Phi_s^j \rangle ds = \langle \xi, \Phi_T^j \rangle - \int_0^T \langle \xi, \sigma^j(r)_t \rangle d\beta_t^j.$$

By (7.7) we have $\sum_j \Phi_T^j = \int_0^T S_{T-t}\sigma(r)_t dW_t$, where the convergence is uniformly on compact time intervals in probability. Thus we obtain

(7.13)

$$\begin{aligned}
&\int_0^T \Big\langle A^* \xi, \sum_j \Phi_s^j \Big\rangle ds = \sum_j \int_0^T \langle A^* \xi, \Phi_s^j \rangle ds \\
&= \sum_j \bigg(\langle \xi, \Phi_T^j \rangle - \int_0^T \langle \xi, \sigma^j(r)_t \rangle d\beta_t^j \bigg) = \Big\langle \xi, \sum_j \Phi_T^j \Big\rangle - \sum_j \int_0^T \langle \xi, \sigma^j(r)_t \rangle d\beta_t^j.
\end{aligned}$$

A similar argumentation as in (7.12), where we apply the classical Fubini theorem, yields

$$\int_0^T \langle A^* \xi, \Pi_s \rangle ds = \langle \xi, \Pi_T \rangle - \int_0^T \langle \xi, \alpha(r)_t \rangle dt. \tag{7.14}$$

Finally, we obtain, by taking into account [31, Lemma VII.4.5(a)] again,

$$\int_0^T \langle A^* \xi, S_s h_0 \rangle ds = \Big\langle \xi, A \int_0^T S_s h_0 ds \Big\rangle = \langle \xi, S_T h_0 \rangle - \langle \xi, h_0 \rangle. \tag{7.15}$$

Inserting (7.12), (7.13), (7.14) and (7.15) into (7.11) shows, since $T \in \mathbb{R}_+$ was arbitrary, that r is a weak solution for (7.1) with $r_0 = h_0$. □

The following result applies in particular if the Hilbert space H is finite dimensional.

7.7. **Lemma.** *Assume $A \in L(H)$, i.e. A is a bounded linear operator, and let $r \in C_{\mathrm{pr}}(H)$ be a weak solution for (7.1) with $r_0 = h_0$. Then, r is also a strong solution for (7.1) with $r_0 = h_0$.*

Proof. Since A is bounded, we have $Ar \in L^1_{\mathrm{loc}}(\lambda; H)$ and $\mathcal{D}(A) = \mathcal{D}(A^*) = H$, implying equation (7.8) for all $\xi \in H$, because $\langle \xi, Ah \rangle = \langle A^* \xi, h \rangle$ for all $\xi, h \in H$. We conclude that r is a strong solution for (7.1) with $r_0 = h_0$. □

8. Existence and uniqueness of mild and weak solutions for stochastic partial differential equations

In this section we introduce the method of the moving frame, which has been announced in the introduction. Loosely speaking we apply a time-dependent coordinate transformation to the SPDE such that "from the moving frame" the SPDE looks like an SDE with appropriately transformed coefficients. The method is in contrast to the point of view, that an SPDE is a PDE together with a non-linear stochastic perturbation. Here we consider an SPDE rather as a time-transformed SDE, where the time transform contains the respective PDE aspect.

We apply this method for an "easy" proof of existence and uniqueness in this general setting. The key argument, which allows to apply the method, is the Szőkefalvi-Nagy theorem, which has been brought to our attention by [17]. We emphasize that in our article we do not need a particular representation of the Hilbert spaces involved in the Szőkefalvi-Nagy theorem (see the subsequent remark). The Szőkefalvi-Nagy theorem is a "ladder", which allows us to "climb" towards several new assertions, but which is not necessary to understand the statements of those assertions.

During this section, we suppose that $\alpha : C_{\mathrm{pr}}(H) \to H_{\mathcal{P}}$, $\sigma : C_{\mathrm{pr}}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\mathrm{pr}}(H) \to H_{\mathcal{P}\otimes\mathcal{E}}$ satisfy Assumptions 3.3 and 3.4. Moreover, we impose the following assumption.

8.1. **Assumption.** *There exists another separable Hilbert space $\mathcal{H}$, a C_0-group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ and continuous linear operators $\ell \in L(H,\mathcal{H})$, $\pi \in L(\mathcal{H},H)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{U_t} & \mathcal{H} \\ \uparrow \ell & & \downarrow \pi \\ H & \xrightarrow{S_t} & H \end{array}$$

commutes for every $t \in \mathbb{R}_+$, that is

$$\pi U_t \ell h = S_t h \quad \text{for all } t \in \mathbb{R}_+ \text{ and } h \in H. \tag{8.1}$$

In particular we see that $\pi \circ \ell = id$.

8.2. **Remark.** *Assumption 8.1 is not only frequently fullfilled, which seems surprising at a first view, but it is also possible to describe the respective Hilbert space $\mathcal{H}$ more precisely. Take for instance a self-adjoint strongly continuous semigroup of contractions S on the complex Hilbert space H, then – as a part of the Szőkefalvi-Nagy theorem – the map $t \mapsto S_{|t|}$ is a strongly continuous, positive definite map, i.e. for all $\psi_1,\dots,\psi_n \in H$ and all real times $t_1,\dots,t_n$ the matrix $\big(\langle S_{|t_i-t_j|}\psi_i,\psi_j\rangle\big)$ is positive definite. A positive definite map with values in bounded linear operators can be considered as characteristic function of a vector-valued measure η taking values in positive operators on H. One can define the Hilbert space $\mathcal{H} = L^2(\mathbb{R},\eta;H)$, i.e. the space of square-integrable H-valued measurable maps f, such that the integral*

$$\int_{\mathbb{R}} \langle f(x), \eta(dx) f(x)\rangle < \infty$$

is finite. H can be embedded via the constant maps $f(x) \equiv h$ for $h \in H$ and $x \in \mathbb{R}$ and the semigroup U is defined via

$$U_t f(x) = \exp(itx) f(x)$$

for $t,x \in \mathbb{R}$. Consequently more precise analysis of the respective generator of S on $\mathcal{H}$ can be performed. Details of the previous considerations and impacts on SPDEs will be presented elsewhere.

According to Proposition 8.5 below, Assumption 8.1 is in particular satisfied if the semigroup $(S_t)_{t\geq 0}$ is pseudo-contractive.

8.3. **Definition.**

(1) *The C_0-semigroup $(S_t)_{t\geq 0}$ is called* contractive *if*

$$\|S_t\| \leq 1, \quad t \geq 0. \tag{8.2}$$

(2) *The C_0-semigroup $(S_t)_{t\geq 0}$ is called* pseudo-contractive *if there exists $\omega \in \mathbb{R}$ such that*

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0. \tag{8.3}$$

8.4. **Remark.** *Sometimes in the literature, e.g., see* [23]*, the notion quasi-contractive is used instead of pseudo-contractive.*

For every C_0-semigroup $(S_t)_{t\geq 0}$ there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S_t\| \leq M e^{\omega t}, \quad t \geq 0 \tag{8.4}$$

see, e.g., [31, Lemma VII.4.2]. Hence, in other words, the semigroup $(S_t)_{t \geq 0}$ is contractive if we can choose $M = 1$ and $\omega = 0$ in (8.4), and it is pseudo-contractive, if we can choose $M = 1$ in (8.4).

Every C_0-semigroup is not far from being pseudo-contractive. Indeed, for an arbitrary $s > 0$, we have, by (8.4), the estimate

$$\|S_t\| \leq e^{\omega(s)t}, \quad t \geq s$$

where we have set $\omega(s) := \frac{\ln M}{s} + \omega$. Nevertheless, there are C_0-semigroups, which are not pseudo-contractive. For a counter example, we choose, following [12, Ex. I.5.7.iii], the Hilbert space $H := L^2(\mathbb{R})$ and the shift semigroup $(S_t)_{t \geq 0}$ with jump, defined as

$$S_t h(x) := \begin{cases} 2h(x+t), & x \in [-t, 0] \\ h(x+t), & \text{otherwise} \end{cases}$$

for $h \in H$. Then $(S_t)_{t \geq 0}$ is a C_0-semigroup on H with $\|S_t\| = 2$ for all $t > 0$, because $\|S_t \mathbb{1}_{[0,t]}\| = 2\|\mathbb{1}_{[0,t]}\|$.

However, many semigroups of practical relevance are pseudo-contractive, for instance η-m-dissipative operators are pseudo-contractive, and then the following result shows that Assumption 8.1 is satisfied.

8.5. **Proposition.** *Assume the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive. Then there exist another separable Hilbert space $\mathcal{H}$ and a C_0-group $(U_t)_{t \in \mathbb{R}}$ on $\mathcal{H}$ such that (8.1) is satisfied, where $\ell \in L(H, \mathcal{H})$ is an isometric embedding and $\pi := \ell^* \in L(\mathcal{H}, H)$ is the orthogonal projection from $\mathcal{H}$ into H.*

Proof. Since the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive, there exists $\omega \in \mathbb{R}$ such that (8.3) is satisfied. Hence, the C_0-semigroup $(T_t)_{t \geq 0}$ defined as $T_t := e^{-\omega t} S_t$, $t \in \mathbb{R}_+$ is contractive. By the Skőkefalvi-Nagy theorem on unitary dilations (see e.g. [29, Thm. I.8.1], or [10, Sec. 7.2]), there exist another separable Hilbert space $\mathcal{H}$ and a unitary C_0-group $(V_t)_{t \in \mathbb{R}}$ in $\mathcal{H}$ such that

$$\pi V_t \ell h = T_t h \quad \text{for all } t \in \mathbb{R}_+ \text{ and } h \in H,$$

where $\ell \in L(H, \mathcal{H})$ is an isometric embedding and the adjoint operator $\pi := \ell^* \in L(\mathcal{H}, H)$ is the orthogonal projection from $\mathcal{H}$ into H. Defining the C_0-group $(U_t)_{t \in \mathbb{R}}$ as $U_t := e^{\omega t} V_t$, $t \in \mathbb{R}$ completes the proof. □

We suppose from now on Assumption 8.1. There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U_t\| \leq M e^{\omega |t|}, \quad t \in \mathbb{R} \tag{8.5}$$

see [12, p. 79]. In order to solve the stochastic partial differential equation (7.1), we consider the $\mathcal{H}$-valued stochastic differential equation

$$\begin{cases} dR_t &= \tilde{\alpha}(R)_t dt + \tilde{\sigma}(R)_t dW_t + \int_E \tilde{\gamma}(R)(t,x)(\mu(dt,dx) - F(dx)dt) \\ R_0 &= h_0, \end{cases} \tag{8.6}$$

where $h_0 : \Omega \to \mathcal{H}$ is an $\mathcal{F}_0$-measurable random variable and where $\tilde{\alpha} : C_{\text{pr}}(\mathcal{H}) \to \mathcal{H}_{\mathcal{P}}$, $\tilde{\sigma} : C_{\text{pr}}(\mathcal{H}) \to L_2(U_0, \mathcal{H})_{\mathcal{P}}$ and $\tilde{\gamma} : C_{\text{pr}}(\mathcal{H}) \to \mathcal{H}_{\mathcal{P} \otimes \mathcal{E}}$ are defined as

$$\tilde{\alpha}(R)_t := U_{-t} \ell \alpha(\pi U R)_t, \tag{8.7}$$

$$\tilde{\sigma}(R)_t := U_{-t} \ell \sigma(\pi U R)_t, \tag{8.8}$$

$$\tilde{\gamma}(R)(t,x) := U_{-t} \ell \gamma(\pi U R)(t,x). \tag{8.9}$$

Note that $\tilde{\alpha}, \tilde{\sigma}, \tilde{\gamma}$ indeed map into the respective spaces of predictable processes, because $(t,h) \mapsto U_t h$ is continuous on $\mathbb{R} \times \mathcal{H}$, see, e.g., [31, Lemma VII.4.3]. They also fulfill Assumptions 3.3 and 3.4, where $L(t)$ is replaced by $\|\ell\| M^2 e^{2\omega t} \|\pi\| L(t)$.

Defining $\tilde{\sigma}^j : C_{\mathrm{pr}}(\mathcal{H}) \to \mathcal{H}_{\mathcal{P}}$ as $\tilde{\sigma}^j(R)_t := \sqrt{\lambda_j}\tilde{\sigma}(R)_t e_j$ for each j, we obtain

$$\tilde{\sigma}^j(R)_t = U_{-t}\ell\sigma^j(\pi U R)_t, \quad t \in \mathbb{R}_+. \tag{8.10}$$

According to Theorem 3.9, for each $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H})$ there exists a unique càdlàg solution $R \in C_{\mathrm{pr}}(\mathcal{H})$ for (8.6) with $R_0 = h_0$ satisfying

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|R_t\|^2\bigg] < \infty \quad \text{for all } T \in \mathbb{R}_+. \tag{8.11}$$

8.6. **Theorem.** *Suppose that Assumptions 3.3, 3.4 and 8.1 are fulfilled. For each $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ there exists a unique càdlàg mild and weak solution $r \in C_{\mathrm{pr}}(H)$ for (7.1) with $r_0 = h_0$ satisfying (3.12), and it is given by $r := \pi U R$, where $R \in C_{\mathrm{pr}}(\mathcal{H})$ denotes the càdlàg solution for (8.6) with $R_0 = \ell h_0$.*

Proof. Let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ be arbitrary. The H-valued process $r := \pi U R$ belongs to $C_{\mathrm{pr}}(H)$, it satisfies (3.12) by virtue of (8.11), and it is also càdlàg, because $(t, h) \mapsto U_t h$ is continuous on $\mathbb{R}_+ \times \mathcal{H}$, see, e.g., [31, Lemma VII.4.3].

Using (7.5), (8.10) and (8.1), we obtain

$$\begin{aligned} r_t &= \pi U_t \ell h_0 + \pi U_t \int_0^t U_{-s}\ell\alpha(\pi U R)_s ds + \pi U_t \sum_j \int_0^t U_{-s}\ell\sigma^j(\pi U R)_s d\beta_s^j \\ &\quad + \pi U_t \int_0^t \int_E U_{-s}\ell\gamma(\pi U R)(s,x)(\mu(ds,dx) - F(dx)ds) \\ &= S_t h_0 + \int_0^t S_{t-s}\alpha(r)_s ds + \sum_j \int_0^t S_{t-s}\sigma^j(r)_s d\beta_s^j \\ &\quad + \int_0^t \int_E S_{t-s}\gamma(r)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0 \end{aligned}$$

showing that r is a mild solution for (7.1) with $r_0 = h_0$. By virtue of Lemma 3.5 we have $\sigma(r) \in L^2(W; L_2^0)$ and $\gamma(r) \in L^2(\mu; H)$. Applying Lemma 7.6 proves that r is also a weak solution for (7.1) with $r_0 = h_0$.

For two càdlàg mild solutions $r, \tilde{r} \in C_{\mathrm{pr}}(H)$ of (7.1) with $r_0 = \tilde{r}_0 = h_0$ and an arbitrary $T \in \mathbb{R}_+$, by using Hölder's inequality, the Itô-isometries (2.1), (2.4) and the Lipschitz conditions (3.5), (3.6), (3.7), the inequality

$$\mathbb{E}[\|r_t - \tilde{r}_t\|^2] \leq 3M^2 e^{2\omega T}(T+2)L(T)^2 \int_0^t \mathbb{E}[\|r_s - \tilde{r}_s\|^2] ds, \quad t \in [0,T]$$

is valid, where $M \geq 1$ and $\omega \in \mathbb{R}$ stem from (8.5). Using the Gronwall Lemma and the hypothesis that r and $\tilde{r}$ are càdlàg, we conclude that r and $\tilde{r}$ are indistinguishable. Taking into account Lemma 7.5, this proves the desired uniqueness of mild and weak solutions for (7.1). □

8.7. **Remarks.**

(1) *The idea to use the Skőkefalvi-Nagy theorem on unitary dilations in order to overcome the difficulties arising from stochastic convolutions, is due to E. Hausenblas and J. Seidler, see* [17] *and* [16].

(2) *If in addition conditions (4.1)–(4.6) are satisfied for some $p \geq 2$, then, for each $h_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, the solution r for (7.1) with $r_0 = h_0$ satisfies the L^p-estimate (4.7).*

9. Stability and Regularity of stochastic partial differential equations

We shall now deal with stability of stochastic partial differential equations of the kind (7.1). As in Section 8, we assume that $\alpha : C_{\rm pr}(H) \to H_{\mathcal{P}}$, $\sigma : C_{\rm pr}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\rm pr}(H) \to H_{\mathcal{P}\otimes\mathcal{E}}$ fulfill Assumptions 3.3 and 3.4. For each $n \in \mathbb{N}$, let $\alpha_n : C_{\rm pr}(H) \to H_{\mathcal{P}}$, $\sigma_n : C_{\rm pr}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma_n : C_{\rm pr}(H) \to H_{\mathcal{P}\otimes\mathcal{E}}$ be such that Assumptions 5.1 and 5.2 are fulfilled. Furthermore, let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and for each $n \in \mathbb{N}$ let $h_0^n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $B_n \in \mathcal{E}$ be given.

We suppose that Assumption 5.4 is fulfilled, where $r \in C_{\rm pr}(H)$ denotes the mild and weak solution for (7.1) with $r_0 = h_0$. Then we have

$$\begin{aligned} C_n(t,r) &:= \|\ell\|^2 M^2 e^{2\omega t} \|\pi\|^2 (\mathbb{E}[\|h_0 - h_0^n\|^2] \\ &\quad + M^2 e^{2\omega t}(t A_n(t,r) + 4\Sigma_n(t,r) + 4\Gamma_n(t,r) + 4G_n(t,r))) \to 0 \end{aligned}$$

as $n \to \infty$ for all $t \in \mathbb{R}_+$, where $A_n(t,r)$, $\Sigma_n(t,r)$, $\Gamma_n(t,r)$ and $G_n(t,r)$ are defined in (5.4)–(5.7).

We suppose that Assumption 8.1 is fulfilled and define $\tilde{\alpha} : C_{\rm pr}(\mathcal{H}) \to \mathcal{H}_{\mathcal{P}}$, $\tilde{\sigma} : C_{\rm pr}(\mathcal{H}) \to L_2(U_0, \mathcal{H})_{\mathcal{P}}$ and $\tilde{\gamma} : C_{\rm pr}(\mathcal{H}) \to \mathcal{H}_{\mathcal{P}\otimes\mathcal{E}}$ by (8.7), (8.8), (8.9). Moreover, for each $n \in \mathbb{N}$, we define $\tilde{\alpha}_n : C_{\rm pr}(\mathcal{H}) \to \mathcal{H}_{\mathcal{P}}$, $\tilde{\sigma}_n : C_{\rm pr}(\mathcal{H}) \to L_2(U_0, \mathcal{H})_{\mathcal{P}}$ and $\tilde{\gamma}_n : C_{\rm pr}(\mathcal{H}) \to \mathcal{H}_{\mathcal{P}\otimes\mathcal{E}}$ as

$$\begin{aligned} \tilde{\alpha}_n(R)_t &:= U_{-t}\ell\alpha_n(\pi U R)_t, \\ \tilde{\sigma}_n(R)_t &:= U_{-t}\ell\sigma_n(\pi U R)_t, \\ \tilde{\gamma}_n(R)(t,x) &:= U_{-t}\ell\gamma_n(\pi U R)(t,x). \end{aligned}$$

According to Theorem 8.6, $r := \pi U R \in C_{\rm pr}(H)$ is the unique càdlàg mild and weak solution for (7.1) with $r_0 = h_0$, where $R \in C_{\rm pr}(\mathcal{H})$ denotes the unique càdlàg solution for (8.6) with $R_0 = \ell h_0$, and, for each $n \in \mathbb{N}$, $r^n := \pi U R^n \in C_{\rm pr}(H)$ is the unique càdlàg mild and weak solution for

$$\begin{cases} dr_t^n &= (Ar_t^n + \alpha(r^n)_t)dt + \sigma(r^n)_t dW_t + \int_E \gamma(r^n)(t,x)(\mu_{B_n}(dt,dx) - F_{B_n}(dx)dt) \\ r_0^n &= h_0^n, \end{cases}$$

where $R^n \in C_{\rm pr}(\mathcal{H})$ denotes the unique càdlàg solution for

$$\begin{cases} dR_t^n &= \tilde{\alpha}(R^n)_t dt + \tilde{\sigma}(R^n)_t dW_t + \int_E \tilde{\gamma}(R^n)(t,x)(\mu_{B_n}(dt,dx) - F_{B_n}(dx)dt) \\ R_0^n &= \ell h_0^n. \end{cases}$$

In the upcoming result, the constants $M \geq 1$ and $\omega \in \mathbb{R}$ appearing in (9.1) and (9.2) stem from the growth estimate (8.5) of the group $(U_t)_{t\in\mathbb{R}}$.

9.1. **Proposition.** *Suppose that Assumptions 3.3, 3.4, 5.1, 5.2, 5.4 and 8.1 are fulfilled. For each $t \in \mathbb{R}_+$ the estimate*

$$\mathbb{E}[\|r_t - r_t^n\|^2] \leq 8e^{8t(t+8)\|\ell\|^2\|M^4 e^{4\omega t}\|\pi\|^2 L(t)^2} C_n(t,r) \to 0 \quad \textit{for } n \to \infty \tag{9.1}$$

is valid, and for every $T \in \mathbb{R}_+$ we have

(9.2)

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{t\in[0,T]} \|r_t - r_t^n\|^2\bigg] \\ &\leq 8\left(1 + 8T(T+8)\|\ell\|^2 M^4 e^{4\omega T}\|\pi\|^2 L(T)^2 e^{8T(T+8)\|\ell\|^2 M^4 e^{4\omega T}\|\pi\|^2 L(T)^2}\right) C_n(T,r) \to 0 \end{aligned}$$

for $n \to \infty$.

Proof. Note that the coefficients $\tilde{\alpha}, \tilde{\sigma}, \tilde{\gamma}$ and $\tilde{\alpha}_n, \tilde{\sigma}_n, \tilde{\gamma}_n$, $n \in \mathbb{N}$ fulfill Assumptions 3.3, 3.4, 5.1 and 5.2, where $L(t)$ is replaced by $\|\ell\| M^2 e^{2\omega t}\|\pi\| L(t)$.

By Assumption 5.4, and since $r = \pi U R$, we have
(9.3)
$$\begin{aligned}\tilde{C}_n(t,R) &:= \mathbb{E}[\|\ell h_0 - \ell h_0^n\|^2] + t\tilde{A}_n(t,R) + 4\tilde{\Sigma}_n(t,R) + 4\tilde{\Gamma}_n(t,R) + 4\tilde{G}_n(t,R) \\ &\leq \|\ell\|^2 \mathbb{E}[\|h_0 - h_0^n\|^2] + \|\ell\|^2 M^2 e^{2\omega t}(tA_n(t,r) + 4\Sigma_n(t,r) + 4\Gamma_n(t,r) + 4G_n(t,r))\end{aligned}$$

converging to zero for $n \to \infty$ for all $t \in \mathbb{R}_+$, where we have set

$$\begin{aligned}\tilde{A}_n(t,R) &:= \mathbb{E}\bigg[\int_0^t \|\tilde{\alpha}(R)_s - \tilde{\alpha}_n(R)_s\|^2 ds\bigg], \\ \tilde{\Sigma}_n(t,R) &:= \mathbb{E}\bigg[\int_0^t \|\tilde{\sigma}(R)_s - \tilde{\sigma}_n(R)_s\|^2_{L_2(U_0,\mathcal{H})} ds\bigg], \\ \tilde{\Gamma}_n(t,R) &:= \mathbb{E}\bigg[\int_0^t \int_E \|\tilde{\gamma}(R)(s,x) - \tilde{\gamma}_n(R)(s,x)\|^2 F(dx) ds\bigg], \\ \tilde{G}_n(t,R) &:= \mathbb{E}\bigg[\int_0^t \int_{E \setminus B_n} \|\tilde{\gamma}(R)(s,x)\|^2 F(dx) ds\bigg].\end{aligned}$$

Applying Proposition 5.5, for each $t \in \mathbb{R}_+$ the estimate

$$\mathbb{E}[\|R_t - R_t^n\|^2] \leq 8 e^{8t(t+8)\|\ell\|^2 M^4 e^{4\omega t} \|\pi\|^2 L(t)^2} \tilde{C}_n(t,R) \to 0 \quad \text{for } n \to \infty$$

is valid, and for every $T \in \mathbb{R}_+$ we have

$$\begin{aligned}&\mathbb{E}\bigg[\sup_{t \in [0,T]} \|R_t - R_t^n\|^2\bigg] \\ &\leq 8\Big(1 + 8T(T+8)\|\ell\|^2 M^4 e^{4\omega T}\|\pi\|^2 L(T)^2 e^{8T(T+8)\|\ell\|^2 M^4 e^{4\omega T}\|\pi\|^2 L(T)^2}\Big) \tilde{C}_n(T,R)\end{aligned}$$

converging to zero for $n \to \infty$. Since $r = \pi U R$ and $r^n = \pi U R^n$ for all $n \in \mathbb{N}$, we arrive, by taking into account (9.3), at the desired estimates (9.1) and (9.2). □

Analogously to stability results also the results on regularity can be transfered to SPDEs by the method of the moving frame. The arguments of Section 6 can be transfered literally. The same arguments hold true for L^p-estimates.

10. Stochastic partial differential equations with state dependent coefficients

In this section, we deal with stochastic partial differential equations with state dependent coefficients, which is a special case of the framework from Section 8.

Let $\alpha : \mathbb{R}_+ \times H \to H$, $\sigma : \mathbb{R}_+ \times H \to L_2^0$ and $\gamma : \mathbb{R}_+ \times H \times E \to H$ be measurable.

10.1. **Assumption.** *We assume that*

$$\begin{aligned}\sup_{t \in [0,T]} \|\alpha(t,0)\| &< \infty, \\ \sup_{t \in [0,T]} \|\sigma(t,0)\|_{L_2^0} &< \infty, \\ \sup_{t \in [0,T]} \int_E \|\gamma(t,0,x)\|^2 F(dx) &< \infty\end{aligned}$$

for every $T \in \mathbb{R}_+$.

10.2. **Assumption.** *We assume that there is a non-decreasing function* $L : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\begin{aligned} \|\alpha(t,h_1) - \alpha(t,h_2)\| &\leq L(t)\|h_1 - h_2\|, \\ \|\sigma(t,h_1) - \sigma(t,h_2)\|_{L_2^0} &\leq L(t)\|h_1 - h_2\|, \\ \bigg(\int_E \|\gamma(t,h_1,x) - \gamma(t,h_2,x)\|^2 F(dx) \bigg)^{\frac{1}{2}} &\leq L(t)\|h_1 - h_2\| \end{aligned}$$

for all $t \in \mathbb{R}_+$ *and all* $h_1, h_2 \in H$.

Notice that Assumption 10.1 is in particular satisfied if $\alpha(\cdot,0)$ and $\sigma(\cdot,0)$ are continuous, and if $\gamma(\cdot,0,x)$ is continuous for every $x \in E$ and there exists a function $\delta \in L^2(E,\mathcal{E},F)$ such that $\|\gamma(t,0,x)\| \leq \delta(x)$ for all $(t,x) \in \mathbb{R}_+ \times E$.

10.3. **Corollary.** *Suppose that Assumptions 10.1, 10.2 and 8.1 are fulfilled. For each random variable* $h_0 \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};H)$*, there exists a unique càdlàg, adapted, mean square continuous mild and weak solution* r *for*

$$\begin{cases} dr_t &= (Ar_t + \alpha(t,r_t))dt + \sigma(t,r_t)dW_t + \int_E \gamma(t,r_{t-},x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &= h_0 \end{cases}$$

satisfying (3.12).

Proof. The maps $\alpha : C_{\text{pr}}(H) \to H_{\mathcal{P}}$, $\sigma : C_{\text{pr}}(H) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\text{pr}}(H) \to H_{\mathcal{P}\otimes\mathcal{E}}$ defined as

$$\begin{aligned} \alpha(r)_t &:= \alpha(t,r_t), \quad t \in \mathbb{R}_+ \\ \sigma(r)_t &:= \sigma(t,r_t), \quad t \in \mathbb{R}_+ \\ \gamma(r)(t,x) &:= \gamma(t,r_t,x), \quad (t,x) \in \mathbb{R}_+ \times E. \end{aligned}$$

are well-defined, by abuse of notation denoted by the same letter, and satisfy conditions Assumptions 3.3 and 3.4. Since $r = r_-$ for every process from $C_{\text{pr}}(H)$, applying Theorem 8.6 yields the assertion. □

We close this section with the time-homogeneous case. Let $\alpha : H \to H$, $\sigma : H \to L_2^0$ and $\gamma : H \times E \to H$ be measurable.

10.4. **Assumption.** *We assume* $\int_E \|\gamma(0,x)\|^2 F(dx) < \infty$.

10.5. **Assumption.** *We assume that there is a constant* $L \geq 0$ *such that*

$$\begin{aligned} \|\alpha(h_1) - \alpha(h_2)\| &\leq L\|h_1 - h_2\|, \\ \|\sigma(h_1) - \sigma(h_2)\|_{L_2^0} &\leq L\|h_1 - h_2\|, \\ \bigg(\int_E \|\gamma(h_1,x) - \gamma(h_2,x)\|^2 F(dx) \bigg)^{\frac{1}{2}} &\leq L\|h_1 - h_2\| \end{aligned}$$

for all $h_1, h_2 \in H$.

10.6. **Corollary.** *Suppose that Assumptions 10.4, 10.5 and 8.1 are fulfilled. For each random variable* $h_0 \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};H)$*, there exists a unique càdlàg, adapted, mean square continuous mild and weak solution* r *for*

$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-},x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &= h_0 \end{cases}$$

satisfying (3.12).

Proof. The statement is a particular case of Corollary 10.3. □

10.7. **Remark.** *The time-inhomogenous case can be considered by an extension of the state space from H to $\mathbb{R} \times H$. However, one has to pay attention at the boundary points of the interval $[0,T]$, where the vector fields have to be extended to the whole real line. Nevertheless we shall consider in the setting of our numerical applications the time-homogenous case as the most characteristic one for all further applications.*

11. High-order (explicit-implicit) numerical schemes for stochastic partial differential equations with weak convergence order

In this section (and only here) we assume that the Wiener process and the Poisson random measure are independent. We sketch in this last section high-order explicit-implicit numerical schemes for stochastic partial differential equations with state-dependent coefficients as in Section 10. By the stability results from Section 9 we can reduce the problem to finitely active Poisson processes and a finite number of Wiener processes. We apply the results of [5] for the time-dependent SDE, which – due to the "method of the moving frame" – can be transfered to the general SPDE case. Our main focus is not to outline all possible schemes (weak schemes, strong schemes, cubature formulations, Taylor formulations), but to describe the basic short-time asymptotic relation and the general analytic conditions on the vector fields in order to obtain a weak convergence order for the high-order schemes.

We apply the respective notions from Section 7, 8 and 10 in order to formulate our conditions on the vector fields. Having the stability results of Section 9 in mind we do assume finite activity of the Poisson random measure, i.e. $F(E) < \infty$.

We consider here SPDEs of the type

(11.1)
$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &\in H \end{cases}$$

Let $T > 0$ denote a time-horizon. As in Section 10 we introduce measurable maps $\alpha : H \to H$, $\sigma : H \to L_2^0$ and $\gamma : H \times E \to H$ and define maps $\tilde{\alpha} : [0,T] \times \mathcal{H} \to \mathcal{H}$, $\tilde{\sigma} : [0,T] \times \mathcal{H} \to \mathcal{H}$ and $\tilde{\gamma} : [0,T] \times \mathcal{H} \times E \to \mathcal{H}$ defined as

$$\tilde{\alpha}(R) := U_{-t}\ell\alpha(\pi U_t R), \quad t \in [0,T]$$
$$\tilde{\sigma}(R) := U_{-t}\ell\sigma(\pi U_t R), \quad t \in [0,T]$$
$$\tilde{\gamma}(R,x) := U_{-t}\ell\gamma(\pi U_t R, x), \quad (t,x) \in [0,T] \times E.$$

are well-defined for $R \in \mathcal{H}$, whence we can formulate the assumptions on those maps.

11.1. **Assumption.** *Fix $m \geq 2$ (a degree of accuracy for the high order scheme) and $T > 0$. We assume that the vector fields α, σ, γ are smooth on H and have bounded support within a ball of radius C_0 such that $\|U_t h\| \leq C$ for $\|h\| \leq C_1$ and $0 \leq t \leq T$. Furthermore we assume that*

$$\sup_{t\in[0,T],\, h\in\mathcal{H}, \|h\|\leq C} \|\partial_t^{k_1}\partial_h^{k_2}\tilde{\alpha}(t,h)\| < \infty,$$
$$\sup_{t\in[0,T],\, h\in\mathcal{H}, \|h\|\leq C} \|\partial_t^{k_1}\partial_h^{k_2}\tilde{\sigma}(t,h)\|_{L_2^0} < \infty,$$
$$\sup_{t\in[0,T],\, h\in\mathcal{H}, \|h\|\leq C} \int_E \|\partial_t^{k_1}\partial_h^{k_2}\tilde{\gamma}(t,h,x)\|^2 F(dx) < \infty$$

holds true for all $k_1 + k_2 \leq m+1$.

11.2. **Remark.** *As outlined in* [30] *the previous assumptions can be summarized as a* $\mathrm{Lip}(m+1)$*-conditions on the corresponding time-dependent vector fields $\tilde{\alpha}, \tilde{\sigma}, \tilde{\gamma}$. This has an important meaning for the extension of our theory towards rough paths.*

11.3. **Remark.** *When we speak of test functions we mean smooth bounded functions $g : \mathcal{H} \to \mathbb{R}$, with all derivatives bounded, such that there is a finite index $k \geq m+1$, an element $\lambda \in \rho(\mathcal{A})$ of the resolvent set of the generator $\mathcal{A}$ of U so that*

$$g \circ (\lambda - A)^k$$

is smooth with all derivatives bounded, too, on $\mathcal{H}$. We also refer to the discussion in [5, Example 4.8, Remark 4.9], *since test functions are quite a delicate issue. Analogously test functions on H are defined.*

We assume now $\gamma = 0$, such that we find ourselves in a pure diffusion case. Furthermore we assume that the driving Wiener noise is finite dimensional, in other words we can write the stochastic partial differential equation in the moving frame and on the extended phase space $\mathbb{R} \times \mathcal{H}$:

$$dR_t = \tilde{\alpha}(s, R_t)dt + \sum_{i=1}^{d} \tilde{\sigma}_i(s, R_t)dW_t^i, \; ds = dt, \tag{11.2}$$

$$R_t = r_0, \; s_0 = 0. \tag{11.3}$$

Let us fix $m \geq 2$. Notice that the Assumption 11.1 implies Assumptions 3.3 and 3.4, in particular we need the vector fields to be $(m+1)$-times differentiable in all variables. We shall first state the result on short-time asymptotics. We apply the notations for iterated stochastic integrals from [5] and do not repeat them here.

11.4. **Theorem.** *Let $g : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ be a smooth test function and let us define vector fields β_i on the extended phase space $H \times [0, T]$ by the following formulas:*

$$\beta_0(R, s) = \big(\tilde{\alpha}(R, s) - \frac{1}{2}\sum_{i=1}^{d} D\tilde{\sigma}_i(R, s) \bullet \tilde{\sigma}_i(R, s), 1\big), \tag{11.4}$$

$$\beta_i(R, s) = \big(\tilde{\sigma}_i(R, s), 0\big), \tag{11.5}$$

for $i = 0, \dots, d$. Then we have the following asymptotic formula,

$$g(R_t, t) = \sum_{\substack{k \leq m, \, (i_1,\dots,i_k) \in \mathcal{A} \\ \deg(i_1,\dots,i_k) \leq m}} (\beta_{i_1} \cdots \beta_{i_k} g)(R_0, 0) W_t^{(i_1,\dots,i_k)} + R_m(t, g, R_0), \quad R_0 \in \mathcal{H}, \tag{11.6}$$

with

$$\sqrt{E(R_m(t, g, R_0)^2)} \leq C t^{\frac{m+1}{2}} \max_{m < \deg(i_1,\dots,i_k) \leq m+2} \; \sup_{0 \leq s \leq t} |E(\beta_{i_1} \cdots \beta_{i_k} g(R_s, s))|. \tag{11.7}$$

Proof. The proof is a direct consequence of the results of [5, Prop. 3.1], where one additionally observes the necessary degrees of differentiability which are needed for the result. Notice in particular that the remainder term stays bounded due to the conditions of Assumption 11.1. □

As explained in the literature, for instance in [5] or [19], we can derive high-order schemes (strong or weak) from the given short time-asymptotic expansion. The weak order of convergence – given a short-time asymptotics of order $t^{\frac{m+1}{2}}$ – is then $t^{\frac{m-1}{2}}$. Therefore we obtain high-order schemes for the time-dependent system (11.2).

11.5. **Corollary.** *Let $f : H \to \mathbb{R}$ be a smooth test function and define $g(R, s) = f(\pi(U_s R))$ for real s and $R \in \mathcal{H}$. Then – assuming the previous notations – we* *have the following asymptotic formula,*

(11.8)
$$f(r_t) = \sum_{\substack{k\leq m,\, (i_1,\ldots,i_k)\in\mathcal{A}\\ \deg(i_1,\ldots,i_k)\leq m}} (\beta_{i_1}\cdots\beta_{i_k} g)(r_0,0) W_t^{(i_1,\ldots,i_k)} + R_m(t,g,r_0), \quad r_0 \in H,$$

with

(11.9)
$$\sqrt{E(R_m(t,g,r_0)^2)} \leq C t^{\frac{m+1}{2}} \max_{m<\deg(i_1,\ldots,i_k)\leq m+2} \sup_{0\leq s\leq t} |E(\beta_{i_1}\cdots\beta_{i_k} f(\pi(U_s R_s)))|.$$

Notice that the terms $(\beta_{i_1}\cdots\beta_{i_k} g)(r_0,0)$ *are well defined on* H *and only involve the semigroup* S *and certain derivatives of* f*. This is the short-time asymptotic expansion for the SPDE* (11.1)

These schemes for the approximate calculation of R_t can be transfered to H by applying $\pi \circ U_t$ to it. Inspecting the resulting scheme on H one easily observes that each appearance of U cancels out or is replaced by the strongly continuous semigroup S. This phenomenon is *no* magic cancelation, but simply the result of inverting an invertible coordinate transform. Hence we have constructed a high-order scheme on H for the original equation, and those schemes correspond precisely to the schemes described in [5], however, the approach is simplified to its essence. Also it is much easier to obtain strong schemes by means of the "moving frame", since the calculations can be performed for an SDE and transfered to the SPDE case. More precisely one can state that *all high order, strong or weak schemes for SDEs can be transfered into the realm of SPDEs* preserving the order of convergence. For instance, the high-order Taylor schemes called N-Euler schemes (see [30] for details), can be transfered in this way. On the other hand the results in [5] are slightly more general, since there the authors do not need to assume that the semigroup is pseudocontractive. What appears through the "method of the moving frame" is an implicit-explicit structure of the scheme. Formulating the scheme for $m = 2$ shows already that the semigroup itself appears in the short-time asymptotics. Similarly higher order schemes like the Milstein-scheme, or cubature schemes, can be formulated.

11.1. **Pure Diffusion case – the weak Euler scheme and cubature schemes.** We formulate the short-time asymptotic formula in the case $m = 2$ by taking the definitions of the vector fields $\beta_0,\ldots,\beta_d$ and a smooth test function $g : \mathcal{H} \to \mathbb{R}$, which does not depend on the additional (time-)state s, then

$$\begin{aligned} g(R_t) =& g(R_0) + \beta_0 g(R_0) t + \sum_{i=1}^d S_t \beta_i g(R_0) W_t^i + \\ & \sum_{i,j=1}^d \beta_i\beta_j g(R_0) W_t^{(i,j)} + \mathcal{O}(t^{\frac{3}{2}}) \\ =& g(R_0) + Dg(R_0)\bullet\alpha(R_0)t + \sum_{i=1}^d Dg(R_0)\bullet\sigma_i(R_0)W_t^i + \\ & + \sum_{i=1}^d Dg(R_0)\bullet(D\sigma_i(R_0)\bullet\sigma_i(R_0))\frac{(W_t^i)^2 - t}{2} + \\ & + \sum_{i\neq j=1}^d Dg(R_0)\bullet(D\sigma_i(R_0)\bullet\sigma_j(R_0))W_t^{(i,j)} + \mathcal{O}(t^{\frac{3}{2}}) \end{aligned}$$

for $t \geq 0$, $R_0 \in \mathcal{H}$ and g being an appropriate test function. Heading for a strong Euler-Maruyama-scheme this yields the first iteration step from $0 \to t$

$$R_0 \mapsto R_0 + \alpha(R_0)t + \sum_{i=1}^{d} \sigma_i W_t^i.$$

For the next step in the iteration we need the asymptotic expansion at time t and therefore also the vector fields $\tilde{\alpha}, \tilde{\sigma}$ appear at time t, namely

$$R_t \mapsto R_t + \tilde{\alpha}(t, R_t)t + \sum_{i=1}^{d} \tilde{\sigma}_i W_t^i.$$

However, when one transfer the iteration of these two steps via $\pi \circ U_{2t}$ to H the described cancelation happens and one obtains the two-fold iteration of the time-homogenous scheme

$$r_0 \mapsto S_t r_0 + S_t \alpha(r_0)t + \sum_{i=1}^{d} S_t \sigma_i(r_0) W_t^i.$$

Notice that this scheme is implicit in the linear PDE-part and explicit in the stochastic components and the non-linear drift component. Notice also that the weak convergence order 1 is obtained if the Assumptions 11.1 for $m = 2$ are satisfied and if test functions f are considered. All cubature schemes, as outlined in [5] can be obtained in this way by showing the (weak) rate of convergence for the SDE on $\mathcal{H}$ and transfering the results to the SPDE on $\mathcal{H}$.

11.2. **Pure Diffusion case – the Kusuoka schemes.** Also the results of [22] can be carried over to the SPDE case, with the significant difference that one does not obtain estimates which are uniform in the state space. These schemes can be considered as weak, high-order Taylor schemes (like the Euler-Maruyama or the Milstein scheme) and the transfer from $\mathcal{H}$ to H brings them into an implicit-explicit form. Precise details will be worked out elsewhere.

11.3. **General Jump Diffusion case.** As in [5] a short time asymptotics for the jump diffusion case is reduced to cases with finitely many jumps on the time interval of consideration, since more jumps only appear with a probability which is negligible with respect to the asymptotic order. One procedure, where the previous short-time asymptotics for pure diffusions is mixed with finitely many jumps, is described in [5] and we refer to it here.

Appendix A. Stochastic Fubini theorem with respect to Poisson measures

In this appendix, we provide a stochastic Fubini theorem with respect to compensated Poisson random measures, see Theorem A.2, which we require for the proof of Lemma 7.6.

We could not find a proof in the literature. In the appendix of [8], it is merely mentioned that it can be provided the same way as in [25], where stochastic integrals with respect to semimartingales are considered. The stochastic Fubini theorem [3, Thm. 5], which is used in the proof of [20, Prop. 5.3], only deals with finite measure spaces.

We start with an auxiliary result.

A.1. **Lemma.** *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$ be two σ-finite measure spaces. We define the product space*

$$(\Omega, \mathcal{F}, \mu) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2).$$

For each $\Phi \in L^2(\Omega, \mathcal{F}, \mu)$ *there exists a sequence*

$$(\Phi_n)_{n\in\mathbb{N}} \subset \operatorname{span}\{\mathbb{1}_{A_1}\mathbb{1}_{A_2} : A_i \in \mathcal{F}_i \text{ with } \mu_i(A_i) < \infty,\ i = 1,2\} \tag{A.1}$$

such that $\Phi_n \to \Phi$ *in* $L^2(\Omega, \mathcal{F}, \mu)$.

Proof. Let $\Phi \in L^2(\Omega, \mathcal{F}, \mu)$ be arbitrary. We decompose $\Phi = \Phi^+ - \Phi^-$ into its positive and negative part. There are sequences $(\Phi_n^+)_{n\in\mathbb{N}}$, $(\Phi_n^-)_{n\in\mathbb{N}}$ of nonnegative measurable functions, taking only a finite number of values, such that $\Phi_n^+ \uparrow \Phi^+$ and $\Phi_n^- \uparrow \Phi^-$, see, e.g., [4, Satz 11.6].

Moreover, since μ_1 and μ_2 are σ-finite measures, there exist sequences $(C_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and $(D_n)_{n\in\mathbb{N}} \subset \mathcal{F}_2$ such that $\mu_1(C_n) < \infty$, $\mu_2(D_n) < \infty$ for all $n \in \mathbb{N}$ and $C_n \uparrow \Omega_1$, $D_n \uparrow \Omega_2$ as $n \to \infty$. By Lebesgue's dominated convergence theorem we have $(\Phi_n^+ - \Phi_n^-)\mathbb{1}_{C_n \times D_n} \to \Phi$ in $L^2(\Omega, \mathcal{F}, \mu)$.

Therefore, we may, without loss of generality, assume that $\Phi = \sum_{j=1}^m c_j \mathbb{1}_{A_j}$, where $m \in \mathbb{N}$, $c_j \in \mathbb{R} \setminus \{0\}$, $j = 1, \dots, m$ and $A_j \in (\mathcal{F}_1 \otimes \mathcal{F}_2) \cap (C_1 \times C_2)$, $j = 1, \dots, m$, where $C_i \in \mathcal{F}_i$, $i = 1,2$ and $\mu_i(C_i) < \infty$, $i = 1,2$.

Note that the trace σ-algebra $(\mathcal{F}_1 \otimes \mathcal{F}_2) \cap (C_1 \times C_2)$ is generated by the algebra

$$\mathcal{A} = \left\{ \biguplus_{k=1}^p D_k \times E_k \,|\, p \in \mathbb{N} \text{ and } D_k \in \mathcal{F}_1 \cap C_1,\ E_k \in \mathcal{F}_2 \cap C_2 \text{ for } k = 1, \dots, p \right\}.$$

By [4, Satz 5.7] there exists, for each $j \in \{1, \dots, m\}$ and each $n \in \mathbb{N}$, a set $B_j^n \in \mathcal{A}$ such that $\mu(A_j \Delta B_j^n) < \frac{1}{m^2 n c_j^2}$.

Setting $\Phi_n := \sum_{j=1}^m c_j \mathbb{1}_{B_j^n}$ for $n \in \mathbb{N}$ we have (A.1) and

$$\begin{aligned}\int_\Omega |\Phi(\omega) - \Phi_n(\omega)|^2 d\mu(\omega) &\leq \int_\Omega \left(\sum_{j=1}^m |c_j| \cdot |\mathbb{1}_{A_j}(\omega) - \mathbb{1}_{B_j^n}(\omega)| \right)^2 d\mu(\omega) \\ &\leq m \sum_{j=1}^m \int_\Omega c_j^2 \mathbb{1}_{A_j \Delta B_j^n}(\omega) d\mu(\omega) = m \sum_{j=1}^m c_j^2 \mu(A_j \Delta B_j^n) < \frac{1}{n}, \quad n \in \mathbb{N}\end{aligned}$$

showing that $\Phi_n \to \Phi$ in $L^2(\Omega, \mathcal{F}, \mu)$. □

Let $T \in \mathbb{R}_+$ be a finite time horizon. In order to have a more convenient notation in the following stochastic Fubini theorem, we introduce the spaces

$$\begin{aligned} L_T^2(\mu) &:= L_T^2(\mu; \mathbb{R}), \\ L_T^2(\lambda) &:= L_T^2([0,T], \mathcal{B}[0,T], \lambda), \\ L_T^2(\mathbb{P} \otimes \lambda) &:= L^2(\Omega \times [0,T], \mathcal{F}_T \otimes \mathcal{B}[0,T], \mathbb{P} \otimes \lambda), \end{aligned}$$

where $L_T^2(\mu; H)$ for a separable Hilbert space H was defined in (2.3), and

$$L_T^p(\mu \otimes \lambda) := L^p(\Omega \times [0,T] \times E \times [0,T], \mathcal{P}_T \otimes \mathcal{E} \otimes \mathcal{B}[0,T], \mathbb{P} \otimes \lambda \otimes F \otimes \lambda) \tag{A.2}$$

for all $p \geq 1$.

A.2. **Theorem.** *For each* $\Phi \in L_T^2(\mu \otimes \lambda)$ *we have*

$$\int_0^T \Phi(\cdot, \cdot, s) ds \in L_T^2(\mu), \tag{A.3}$$

there exists $\phi \in L_T^2(\mathbb{P} \otimes \lambda)$ *such that for* λ*-almost all* $s \in [0,T]$

$$\phi(s) = \int_0^T \int_E \Phi(t,x,s)(\mu(dt,dx) - F(dx)dt) \quad \text{in } L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \tag{A.4}$$

and we have the identity

(A.5)
$$\int_0^T \phi(s)ds = \int_0^T \int_E \left(\int_0^T \Phi(t,x,s)ds \right) (\mu(dt,dx) - F(dx)dt) \quad \textit{in } L^2(\Omega, \mathcal{F}_T, \mathbb{P}).$$

Proof. Let $V \subset L_T^2(\mu \otimes \lambda)$ be the vector space

$$V := \operatorname{span}\{ Kf \mid K \in L_T^2(\mu), f \in L_T^2(\lambda) \}.$$

Let $\Phi \in V$ be arbitrary. Then there exist $n \in \mathbb{N}$ and $c_i \in \mathbb{R}$, $K_i \in L_T^2(\mu)$, $f_i \in L_T^2(\lambda)$, $i = 1, \dots, n$ such that $\Phi = \sum_{i=1}^n c_i K_i f_i$. Moreover we have

$$\begin{aligned} \phi &:= \int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \\ &= \sum_{i=1}^n c_i f_i(\cdot) \int_0^T \int_E K_i(t,x)(\mu(dt,dx) - F(dx)dt) \in L_T^2(\mathbb{P} \otimes \lambda), \\ \int_0^T \Phi(\cdot,\cdot,s)ds &= \sum_{i=1}^n c_i K_i(\cdot,\cdot) \int_0^T f_i(s)ds \in L_T^2(\mu) \end{aligned}$$

and identity (A.5) is valid.

For each $\Phi \in L_T^2(\mu \otimes \lambda) \cap L_T^1(\mu \otimes \lambda)$ we have, according to [18, Prop. II.1.14],

(A.6)
$$\begin{aligned} &\int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \\ &= \sum_{n \in \mathbb{N}} \Phi(\tau_n, \beta_{\tau_n}, \cdot) \mathbb{1}_{\{\tau_n \le T\}} - \int_0^T \int_E \Phi(t,x,\cdot)F(dx)dt, \end{aligned}$$

where $(\tau_n)_{n \in \mathbb{N}}$ is a sequence of stopping times and β denotes an E-valued optional process. By the classical Fubini theorem we deduce that the stochastic integral in (A.6) is $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable. Using the Itô-isometry (2.4) we obtain

$$\begin{aligned} &\int_0^T \mathbb{E} \left[\left(\int_0^T \int_E \Phi(t,x,s)(\mu(dt,dx) - F(dx)dt) \right)^2 \right] ds \\ &= \int_0^T \mathbb{E} \bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 F(dx)dt \bigg] ds < \infty, \end{aligned}$$

because $\Phi \in L_T^2(\mu \otimes \lambda)$ by hypothesis, and we conclude

(A.7)
$$\int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \in L_T^2(\mathbb{P} \otimes \lambda), \quad \Phi \in L_T^2(\mu \otimes \lambda) \cap L_T^1(\mu \otimes \lambda).$$

Now let $\Phi \in L_T^2(\mu \otimes \lambda)$ be arbitrary. By the classical Fubini theorem the integral appearing in (A.3) is $\mathcal{P}_T \otimes \mathcal{E}$-measurable. Hölder's inequality and the hypothesis $\Phi \in L_T^2(\mu \otimes \lambda)$ yield

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \int_E \left(\int_0^T \Phi(t,x,s)ds \right)^2 F(dx)dt \right] \\ &\le T \mathbb{E} \bigg[\int_0^T \int_E \int_0^T |\Phi(t,x,s)|^2 ds F(dx)dt \bigg] < \infty, \end{aligned}$$

and hence (A.3) is valid.

Since the measure F is σ-finite, there exists a sequence $(B_n)_{n\in\mathbb{N}} \subset E$ with $F(B_n) < \infty$, $n \in \mathbb{N}$ and $B_n \uparrow E$. We define

$$\phi_n := \int_0^T \int_E \Phi(t,x,\cdot)\mathbb{1}_{B_n}(x)(\mu(dt,dx) - F(dx)dt), \quad n \in \mathbb{N}.$$

By (A.7) we have $\phi_n \in L^2_T(\mathbb{P}\otimes\lambda)$ for all $n \in \mathbb{N}$. Now, we shall prove that $(\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2_T(\mathbb{P}\otimes\lambda)$.

Let $\epsilon > 0$ be arbitrary. By Lebesgue's theorem, there exists an index $n_0 \in \mathbb{N}$ such that

$$\int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{E\setminus B_n}(x)F(dx)dt\bigg]ds < \epsilon, \quad n \geq n_0$$

For all $m > n \geq n_0$ we obtain by the Itô-isometry (2.4)

$$\int_0^T \mathbb{E}[|\phi_n(s) - \phi_m(s)|^2]ds = \int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{B_m\setminus B_n}(x)F(dx)dt\bigg]ds$$
$$\leq \int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{E\setminus B_n}(x)F(dx)dt\bigg]ds < \epsilon,$$

establishing that $(\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2_T(\mathbb{P}\otimes\lambda)$. Thus, there exists $\phi \in L^2_T(\mathbb{P}\otimes\lambda)$ such that $\phi_n \to \phi$ in $L^2_T(\mathbb{P}\otimes\lambda)$. The relation

$$\int_0^T \mathbb{E}[|\phi_n(s) - \phi(s)|^2]ds \to 0 \quad \text{for } n \to \infty$$

implies that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\mathbb{E}[|\phi_{n_k}(s) - \phi(s)|^2] \to 0 \quad \text{for } \lambda\text{-almost all } s \in [0,T],$$

that is $\phi_{n_k}(s) \to \phi(s)$ in $L^2_T(\Omega, \mathcal{F}_T, \mathbb{P})$ for λ-almost all $s \in [0,T]$. We define

$$\psi := \int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt).$$

By the classical Fubini theorem we have $\Phi(\cdot,\cdot,s) \in L^2_T(\mu)$ for λ-almost all $s \in [0,T]$. The Itô-isometry (2.4) and Lebesgue's theorem yield

$$\mathbb{E}[|\psi(s) - \phi_n(s)|^2] = \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{E\setminus B_n}(x)F(dx)dt\bigg] \to 0 \quad \text{for } n \to \infty,$$

implying $\phi_n(s) \to \psi(s)$ in $L^2_T(\Omega, \mathcal{F}_T, \mathbb{P})$ for λ-almost all $s \in [0,T]$. We infer that $\phi(s) = \psi(s)$ in $L^2_T(\Omega, \mathcal{F}_T, \mathbb{P})$ for λ-almost all $s \in [0,T]$, proving (A.4).

According to Lemma A.1 there exists a sequence $(\Phi_n)_{n\in\mathbb{N}} \subset V$ such that $\Phi_n \to \Phi$ in $L^2_T(\mu\otimes\lambda)$. From the beginning of the proof we know that for each $n \in \mathbb{N}$ we have

$$\int_0^T \Phi_n(\cdot,\cdot,s)ds \in L^2_T(\mu), \quad \int_0^T \int_E \Phi_n(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \in L^2_T(\mathbb{P}\otimes\lambda)$$

and the identity

$$\begin{aligned}&\int_0^T \bigg(\int_0^T \int_E \Phi_n(t,x,s)(\mu(dt,dx) - F(dx)dt)\bigg)ds \\ &= \int_0^T \int_E \bigg(\int_0^T \Phi_n(t,x,s)ds\bigg)(\mu(dt,dx) - F(dx)dt)\end{aligned} \tag{A.8}$$

in $L^2_T(\Omega, \mathcal{F}_T, \mathbb{P})$. By Hölder's inequality, (A.4), the Itô-isometry (2.4) and the convergence $\Phi_n \to \Phi$ in $L^2_T(\mu \otimes \lambda)$ we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left(\int_0^T \int_E \Phi_n(t,x,s)(\mu(dt,dx) - F(dx)dt)\right)ds - \int_0^T \phi(s)ds\right)^2\right] \\
&\leq T \int_0^T \mathbb{E}\left[\left(\int_0^T \int_E \Phi_n(t,x,s)(\mu(dt,dx) - F(dx)dt) - \phi(s)\right)^2\right] ds \qquad \text{(A.9)} \\
&= T \int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi_n(t,x,s) - \Phi(t,x,s)|^2 F(dx)dt\bigg] ds \to 0.
\end{aligned}$$

The Itô-isometry (2.4), Hölder's inequality and the convergence $\Phi_n \to \Phi$ in $L^2_T(\mu \otimes \lambda)$ yield

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_0^T \int_E \left(\int_0^T \Phi_n(t,x,s)ds\right)(\mu(dt,dx) - F(dx)dt) \\
&\quad - \int_0^T \int_E \left(\int_0^T \Phi(t,x,s)ds\right)(\mu(dt,dx) - F(dx)dt)\bigg)^2\bigg] \qquad \text{(A.10)} \\
&= \mathbb{E}\left[\int_0^T \int_E \left(\int_0^T (\Phi_n(t,x,s) - \Phi(t,x,s))ds\right)^2 F(dx)dt\right] \\
&\leq T\mathbb{E}\bigg[\int_0^T \int_E \int_0^T |\Phi_n(t,x,s) - \Phi(t,x,s)|^2 F(dx)dtds\bigg] \to 0.
\end{aligned}$$

Combining (A.8), (A.9) and (A.10) we arrive at (A.5). □

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