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ON STRONG SOLUTIONS FOR POSITIVE DEFINITE JUMP–DIFFUSIONS

EBERHARD MAYERHOFER, OLIVER PFAFFEL, AND ROBERT STELZER

Abstract. We show the existence of unique global strong solutions of a class of stochastic differential equations on the cone of symmetric positive definite matrices. Our result includes affine diffusion processes and therefore extends considerably the known statements concerning Wishart processes.

1. Introduction

A result of the general theory for affine Markov processes on the cone S_d^+ of symmetric positive semi-definite matrices developed in [10] is that for a $d \times d$ matrix-valued Brownian motion B, $d \times d$ matrices Q and β, a symmetric constant drift b, and a positive linear drift $\Gamma : S_d^+ \to S_d^+$, *weak global solutions* exist to the stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= \sqrt{X_t} dB_t Q + Q^\top dB_t^T \sqrt{X_t} + (X_t \beta + \beta^\top X_t + \Gamma(X_t) + b) dt, \\ X_0 &= x \in S_d^+, \end{aligned} \tag{1.1}$$

whenever $b - (d-1)Q^\top Q \in S_d^+$. Above $\sqrt{X}$ denotes the unique positive semi-definite square root of a matrix $X \in S_d^+$. This recent result considerably extends the findings of the fundamental paper by Marie-France Bru [6, Theorem 2] concerning Wishart processes (see also [16, 17, 22], for instance, and references therein), which are a strict subclass of that of affine processes on S_d^+.

In this paper, we focus on the existence of global *strong solutions* of (1.1) and generalizations of it including jumps. Because of the non-Lipschitz diffusion at the boundary of the cone, this problem is a quite delicate one – a-priori it is only clear that a unique local solution of (1.1) exists until X_t hits the boundary of S_d^+, since the SDE is locally Lipschitz in the interior of S_d^+. Furthermore, known results for path-wise uniqueness, for instance that of the seminal paper of Yamada and Watanabe [38, Corollary 3], are essentially one-dimensional, and therefore do not apply. Hence, the present setting seems to be more complicated than the canonical affine one (concerning diffusions on $\mathbb{R}_+^m \times \mathbb{R}^n$, [19, Lemma 8.2]).

Bru's paper [6] also contains results on strong solutions for Wishart processes (see our upcoming Theorem 2.1 and Remark 4.5), however, they are derived under strong parametric restrictions, because her method requires an application of Girsanov's theorem. The latter is based on a martingale criterion, which in the matrix valued setting seems hard to verify. Also, the general result (with a non-vanishing linear drift) only holds until the first time the eigenvalues of the process collide.

However, in this paper we do not limit ourselves to the analysis of (1.1). Instead we consider a similar SDE allowing for a general (not necessarily linear) drift Γ and an additional jump part of finite variation. This implies that many Lévy-driven SDEs on S_d^+ like the positive semi-definite Ornstein-Uhlenbeck (OU) type processes (see [3, 33]) or the volatility process of a multivariate COGARCH process (see [36]), where the existence of global strong solutions has previously been shown by path-wise arguments, are special cases of our setting. Thus our results allow to consider mixtures of such jump processes and Wishart diffusions in applications.

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Programme of the paper. In the subsequent section we state the main result, Theorem 2.2, and relate it to Bru's work which is recalled in Theorem 2.1.

Our method relies on a generalized version of the so-called *McKean's argument*, but avoids the use of Girsanov's theorem. In Section 3 we provide a self-contained proof of a general version of *McKean's argument*, which relies on subtle results concerning local martingales on stochastic intervals. The proof of Theorem 2.2 is delivered in Section 4. We conclude the paper with some final remarks in Section 5.

Applications. Positive semi-definite matrix valued processes are increasingly used in finance, particularly for stochastic modeling of multivariate stochastic volatility phenomena in equity and fixed income models, see [7, 8, 11, 12, 13, 14, 20, 21, 23, 32]. See also [10] and the references therein. The papers mentioned mostly use Bru's class of Wishart diffusions, as this results in multivariate analogues of the popular Heston stochastic volatility model and its extensions, Ornstein-Uhlenbeck type processes ([32]) giving a multivariate generalization of the popular model of [2] or a combination of both ([26]). This motivated a recent revival of the research on affine processes (see [10, 18]). Appropriate multivariate models are especially important for issues like portfolio optimization, portfolio risk management and the pricing of options depending on several underlyings, which are h! eavily influenced by the dependence structure.

Clearly S_d^+-valued processes model the covariances, not the correlations, which are, however, preferable when interpreting the dependence structure. The results of this paper are particularly relevant, when one wants to derive correlation dynamics (see e.g., [7, 8]), because one needs to assume boundary non-attainment conditions for a rigorous derivation.

2. Notation and statement of the main results

Let $d \in \mathbb{N}$ denote the dimension. In this paper we use the following notation, definitions and setting:

- $\mathbb{R}_+ := [0,\infty)$, M_d is the set of real valued $d \times d$ matrices and I_d is the unit matrix.
- $S_d \subset M_d$ is the space of symmetric matrices, and $S_d^+ \subset S_d$ is the cone of symmetric positive semi-definite matrices in S_d and S_d^{++} its interior, i.e. the positive definite matrices. The partial order on S_d induced by the cone is denoted by $\preceq$, and $x \succ 0$, if and only if $x \in S_d^{++}$. We endow S_d with the scalar product $\langle x, y\rangle := \mathrm{Tr}(xy)$, where $\mathrm{Tr}(A)$ denotes the trace of $A \in M_d$. $\| \, \|$ denotes the associated norm, and $d(x, \partial S_d^+)$ is the distance of $x \in S_d^+$ to the boundary ∂S_d^+.
- We are given an appropriate filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$, in short Ω, satisfying the usual hypotheses (complete and right-continuous filtration). B is a $d \times d$ standard Brownian motion on Ω.
- A function $f : S_d^{++} \to U$ with U being (a subset of) a normed space is called *locally Lipschitz* if $\|f(x) - f(y)\| \le K(C)\|x - y\| \,\forall\, x, y \in C$ for all compacts $C \subset U$. f is said to have *linear growth* if $\|f(x)\|^2 \le K(1 + \|x\|^2) \,\forall\, x \in S_d^{++}$.
- An S_d-valued càdlàg adapted stochastic process X is called S_d^+-increasing, if $X_t \succeq X_s$ a.s. for all $t > s \ge 0$. Such a process is necessarily of finite variation on compacts by [3, Lemma 5.21] and hence a semimartingale. We call it of *pure jump type* provided $X_t = X_0 + \sum_{0<s\le t} \Delta X_s$, where $\Delta X_s = X_s - X_{s-}$.

For the necessary background on stochastic analysis we refer to one of the standard references like [25, 34, 35].

In order to illustrate the context of our result, we discuss first Bru's statement concerning strong solutions of affine matrix valued diffusions. For $Q \in M_d$, $\delta \ge d-1$, and $\beta \in M_d$ a Wishart process is the strong solution of the equation

$$\begin{aligned} dX_t &= \sqrt{X_t}dB_tQ + Q^\top dB_t^T\sqrt{X_t} + (X_t\beta + \beta^\top X_t + \delta Q^\top Q)dt, \\ X_0 &= x \in S_d^{++}, \end{aligned} \tag{2.1}$$

on the maximal stochastic interval $[0, T_x)$, where T_x is naturally defined as

$$T_x = \inf\{t > 0 \,:\, X_t \in \partial S_d^+\}.$$

That such a unique local strong solution which does not explode before or at time T_x exists, follows from standard SDE theory, since all the coefficients in (2.1) are locally Lipschitz and of linear growth on S_d^{++}. To be more precise, this follows by appropriately localizing the usual results as e.g. in [34, Chapter V] or by variations of the proofs in [29, Chapter 3]. A localization procedure adapted particularly to certain convex sets like S_d^+ is presented in detail in [37, Section 6.7].

The following is a summary of the results [6, Theorem 2, 2' and 2"]:

Theorem 2.1. *Let $\delta \geq d+1$.*

(i) *If $Q = I_d$ and $\beta = 0$, then $T_x = \infty$.*
(ii) *If $Q \in S_d^{++}$, $-\beta \in S_d^+$ such that β and Q commute, then there exists a solution $(X_t)_{t\in\mathbb{R}_+}$ of (2.1) until the first time τ_x the eigenvalues of X_t collide.*
(iii) *If $\beta = \beta_0 I_d$ and $Q = \gamma I_d$, where $\beta_0, \gamma \in \mathbb{R}$, then $T_x = \infty$ for the solution of $(X_t)_{t\in\mathbb{R}_+}$ of (2.1).*

Consequently, for the respective choice of parameters, there exist unique global strong solutions of the SDE (2.1) on S_d^{++} on $[0,\tau_x)$ resp. on all of $[0,\infty)$.

The aim of this paper is the proof of the following very general result concerning non-attainment of the boundary of S_d^+ and the existence of a unique global strong solution for a generalization of the SDE (1.1).

Theorem 2.2. *Let $b \in S_d$, $Q \in M_d$, $\beta \in M_d$, and let*

- *J be an S_d-valued càdlàg adapted process which is S_d^+-increasing and of pure jump type,*
- *$\Gamma : S_d^{++} \to S_d^+$ be a locally Lipschitz function of linear growth and*
- *Let $g : S_d^{++} \to \mathbb{R}_+$ be a locally Lipschitz function of linear growth.*

If $b \succeq (d+1)Q^TQ$, then the SDE

$$\begin{aligned} dX_t &= \sqrt{X_{t-}}dB_tQ + Q^\top dB_t^T\sqrt{X_{t-}} + (X_{t-}\beta + \beta^\top X_{t-} + \Gamma(X_{t-}) + b)dt + g(X_{t-})dJ_t, \\ X_0 &= x \in S_d^{++}, \end{aligned} \tag{2.2}$$

has a unique adapted càdlàg global strong solution $(X_t)_{t\in\mathbb{R}_+}$ on S_d^{++}. In particular we have $T_x := \inf\{t \geq 0 : X_{t-} \in \partial S_d^+ \text{ or } X_t \notin S_d^{++}\} = \inf\{t \geq 0 : X_{t-} \in \partial S_d^+\} = \infty$ almost surely.

Remark 2.3. (i) *By choosing Γ linear and $J = 0$, we obtain a result for (1.1) which considerably generalizes Theorem 2.1.*
(ii) *A possible choice for J is a matrix subordinator without drift (see [1]), i.e. an S_d^+-increasing Lévy process. By choosing $\Gamma \neq 0$ in (2.2) appropriately our results also apply to SDEs involving matrix subordinators with a non-vanishing drift.*
(iii) *Setting $Q = 0$, $\Gamma = 0$, $g = I_d$ and b equal to the drift of the matrix subordinator, equation (2.2) becomes the SDE of a positive definite OU type process,* [3, 33]. *Likewise, it is straightforward to see that the SDE of the volatility process Y of the multivariate COGARCH process of* [36] *is a special case of (2.2).*
(iv) *An OU–type process on the positive semi-definite matrices is necessarily driven by a Lévy process of finite variation having positive semi-definite jumps only (follows by slightly adapting the arguments in the proof of* [33, Theorem 4.9]*). This entails that a generalization of the above result to a more general jump behavior requires additional technical restrictions.*

3. McKean's argument

The main result of this section is Proposition 3.4 which generalizes an argument of [28, p. 47, Problem 7] concerning continuous local martingales on stochastic intervals used, for instance, in [5, 6, 30]. We keep the tradition of referring to it as *McKean's argument*. It shall be used in Section 4 for a proof of Theorem 2.2. Also, Corollary 3.5 constitutes the original statement for continuous local martingales.

McKean's argument relies on subtle arguments from stochastic analysis, and to the best of our knowledge, a rigorous proof of the original argument is not found in the literature. Therefore, we find it advisable to provide a self-contained proof of our generalization in this section. To start with, we collect some basic definitions regarding stochastic processes defined on stochastic intervals by following mainly [27].

Definition 3.1. *Let $A \in \mathcal{F}$ and let T be a stopping time.*

- *A random variable X on A is a mapping $A \to \mathbb{R}$ which is measurable with respect to the σ-algebra $A \cap \mathcal{F}$.*
- *A family $(X_t)_{t\in\mathbb{R}_+}$ of random variables on $\{t < T\}$ is called a stochastic process on $[0,T)$. If X_t is $\{t < T\} \cap \mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$, then X is said to be adapted.*
- *An adapted process M on $[0,T)$ is called a continuous local martingale on the interval $[0,T)$ if there exists an increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ and a sequence of continuous martingales $(M^{(n)})_{n\in\mathbb{N}}$ (in the usual sense on $[0,\infty)$) such that $\lim_{n\to\infty} T_n = T$ a.s. and $M_t = M_t^{(n)}$ on $\{t < T_n\}$. Other local properties for adapted processes on $[0,T)$ are defined likewise.*
- *A semimartingale on $[0,T)$ is the sum of a càdlàg local martingale on $[0,T)$ and an adapted càdlàg process of locally finite variation on $[0,T)$.*
- *For a continuous local martingale on $[0,T)$ the quadratic variation is the $\mathbb{R}\cup\{\infty\}$-valued stochastic process $[M,M]$ defined by*
$$[M,M]_t = \sup_{n\in\mathbb{N}} [M^{(n)}, M^{(n)}]_{t\wedge T_n} \text{ for all } t \in \mathbb{R}_+.$$

The following is a basic fact concerning local martingales [27, Th. 3.5]:

Lemma 3.2. *Let M be a continuous local martingale on the interval $[0,T)$. Then the following equivalence holds on $\{T > 0\}$ a.s.:*
$$M_{T-}(\omega) := \lim_{t\nearrow T(\omega)} M_t(\omega) \text{ exists in } \mathbb{R} \iff [M,M]_T(\omega) < \infty.$$

Recall that for a (not necessarily finite) stopping time T the stopped σ-algebra is given by
$$\mathcal{F}_T := \{A \in \mathcal{F}_\infty \;:\; A \cap \{T \le t\} \in \mathcal{F}_t, \text{ for all } t \in [0,\infty]\},$$
where $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t\right)$ (cf. [35, p. 43]).

An enlargement of a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$ is a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}, (\widetilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+})$ together with a surjective map $\pi : \widetilde{\Omega} \to \Omega$ satisfying $\pi^{-1}(\mathcal{F}_t) \subseteq \widetilde{\mathcal{F}}_t$ for all $t \in \mathbb{R}_+$, $\pi^{-1}(\mathcal{F}) \subseteq \widetilde{\mathcal{F}}$ and $P = \widetilde{P} \circ \pi^{-1}$. A random variable X on the initial probability space is extended to the enlargement by setting $X(\tilde{\omega}) = X(\pi(\tilde{\omega}))$ (see [35, p. 182]) and again denoted by X. In the following we shall use an enlargement $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}, (\widetilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+})$ of the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_{T_t})_{t\in\mathbb{R}_+})$ which satisfies the usual hypotheses and sup! ports a Brownian motion B independent of T and M. Such an enlargement may be obtained by taking the product of $(\Omega, \mathcal{F}, P, (\mathcal{F}_{T_t})_{t\in\mathbb{R}_+})$ and any probability space supporting a Brownian motion (like in the proof of [35, Theorem 1.7, Chapter V]).

The following statement [27, Th. 3.5] is a version of the *Dambis-Dubins-Schwarz Theorem* [35, Chapter V, Theorem 1.7] on stochastic intervals:

Proposition 3.3. *Let M be a continuous local martingale on the interval $[0,T)$ with $M_0 = 0$ and define for $t \in \mathbb{R}_+$ the stopping time $T_t = \inf\{s \in \mathbb{R}_+ : [M,M]_s > t\}$, where we use the convention that the infimum of an empty set is infinity. Then $W = (W_t)_{t\in\mathbb{R}_+}$ defined by*
$$W_t = \begin{cases} M_{T_t} & \text{if } t < [M,M]_T \\ M_{T-} I_{\{T>0\}} + B_t - B_{[M,M]_T} & \text{if } t \ge [M,M]_T \end{cases}$$
is a Brownian motion on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}, (\widetilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+})$.

This result motivates also to call a continuous local martingale β on $[0,T)$ a Brownian motion on $[0,T)$, if $[\beta,\beta]_t = t$ on $[0,T)$.

Now we are prepared to state and prove our generalized *McKean's argument*:

Proposition 3.4. *Let $Z = (Z_s)_{s\in\mathbb{R}_+}$ be an adapted càdlàg real-valued stochastic process on a stochastic interval $[0, \tau_0)$ such that $Z_0 > 0$ a.s. and $\tau_0 := \inf\{s \geq 0 : Z_{s-} = 0\}$. Suppose $h : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ satisfies the following:*

(i) *For all $t \in [0, \tau_0)$, we have $h(Z_t) = h(Z_0) + M_t + P_t$, where*
  (a) P is an adapted càdlàg locally bounded non-negative process on $[0, \tau_0)$,
  (b) M is a continuous local martingale on $[0, \tau_0)$ with $M_0 = 0$,

(ii) *and $\lim_{z\downarrow 0} h(z) = -\infty$.*

Then $\tau_0 = \infty$ a.s.

Proof. Obviously, $\tau_0 = \inf\{s \in \mathbb{R}_+ : M_{s-} = -\infty\}$ due the non-negativity and local boundedness of P and $\tau_0 > 0$ due to the right continuity. Assume, by contradiction, that $\tau_0 < \infty$ on a set $A \in \mathcal{F}$ with $P(A) > 0$. Since $\lim_{t\nearrow\tau_0} M_t = -\infty$ on A, Lemma 3.2 implies that $[M, M]_{\tau_0} = \infty$ on some set $\bar{A} \subseteq A$ with $P(\bar{A}) = P(A) > 0$. Consider T_t and W_t as defined in Proposition 3.3. Then on $\widetilde{A} := \pi^{-1}(\bar{A})$ we have $W_t = M_{T_t}$ for all $t \in \mathbb{R}_+$ and by construction $\widetilde{P}(\widetilde{A}) = P(\bar{A}) > 0$. Hence, on $\widetilde{A}$ we have that

$$\lim_{t\to\infty} W_t = \lim_{t\to\infty} M_{T_t} = \lim_{t\to\tau_0} M_t = -\infty.$$

But this is impossible, since W is a Brownian motion and the paths of Brownian motion a.s. have infinitely many oscillations and therefore do not diverge to $-\infty$ on a set of positive measure (see e.g. [35, Corollary 1.12, Chapter II]). □

If P is an increasing process, then Proposition 3.4 provides a result concerning submartingales Z. Therefore, the usual McKean's argument for local martingales takes the following form:

Corollary 3.5. *Let $Z = (Z_s)_{s\in\mathbb{R}_+}$ be a real-valued continuous stochastic process on a stochastic interval $[0, \tau_0)$ such that $Z_0 > 0$ a.s. and*
$\tau_0 := \inf\{s \geq 0 : Z_s = 0\}$. Suppose $h : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ satisfies the following:

(i) *$h(Z)$ is a continuous local martingale on $[0, \tau_0)$,*

(ii) *and $\lim_{z\downarrow 0} h(z) = -\infty$ (resp. $+\infty$).*

Then $\tau_0 = \infty$ a.s.

4. Proof of Theorem 2.2

Before we provide a proof of Theorem 2.2, we recall some elementary identities from matrix calculus and provide some further technical lemmata. For a differentiable function $f : M_d \to \mathbb{R}$, we denote by ∇f the usual Frechet (or total) derivative written in coordinates as $(\frac{\partial f}{\partial x_{ij}})_{ij}$.

Lemma 4.1. *On S_d^{++}, we have*

(i) $\nabla \det(x) = \det(x)(x^{-1})$

(ii) $\frac{\partial^2}{\partial x_{ij}\partial x_{kl}} \det(x) = \det(x)[(x^{-1})_{kl}(x^{-1})_{ij} - (x^{-1})_{ik}(x^{-1})_{lj}]$

For a semimartingale X we denote by X^c as usual its continuous part. All semimartingales in the following will have a discontinuous part of finite variation, i.e. $\sum_{0<s\leq t} \|\Delta X_s\|$ is finite for all $t \in \mathbb{R}^+$. Thus we define $X_t^c = X_t - \sum_{0<s\leq t} \Delta X_s$ and note that the quadratic variation of a semimartingale is the one of its local continuous martingale part plus the sum of its squared jumps.

The continuous quadratic variation of X solving (2.2) is only influenced by the Brownian terms and, hence, we have as in [6, Equation (2.4)]:

Lemma 4.2. *Consider the solution X_t of (2.2) on $[0, T_x)$. Then*

$$d[X_{ij}, X_{kl}]_t^c = \left(X_{t-,ik}(Q^\top Q)_{jl} + X_{t-,il}(Q^\top Q)_{jk} + X_{t-,jk}(Q^\top Q)_{il} + X_{t-,jl}(Q^\top Q)_{ik}\right) dt$$

Moreover, we shall need the following result:

Lemma 4.3. *Let X_t be a continuous S_d^+-valued adapted càdlàg stochastic process on a stochastic interval $[0,T)$ with T being a predictable stopping time and let $h : M_d \to M_d$. Then there exists a one-dimensional Brownian motion β^h on $[0,T)$ such that*

$$\operatorname{Tr}\left(\int_0^t h(X_{u-})dB_u\right) = \int_0^t \sqrt{\operatorname{Tr}(h(X_{u-})^\top h(X_{u-}))}d\beta_u^h \tag{4.1}$$

holds on $[0,T)$.

Proof. We define for $t \in [0,T)$,

$$\beta_t^h := \sum_{i,j=1}^d \int_0^t \frac{h(X_{u-})_{ij}}{\sqrt{\operatorname{Tr}(h(X_{u-})^\top h(X_{u-}))}} dB_{t,ji},$$

and since the numerator equals zero, whenever the denominator vanishes, we use the convention $\frac{0}{0} = 1$. Clearly for each i,j and for all $u \in [0,T)$ we have

$$\left| \frac{h(X_{u-})_{,ij}}{\sqrt{\operatorname{Tr}(h(X_{u-})^\top h(X_{u-}))}} \right| \leq 1$$

which ensures that β^h is well-defined, square-integrable and a continuous local martingale on $[0,T)$ by stopping at a sequence of stopping times announcing T. Furthermore, by construction

$$[\beta^h, \beta^h]_t = \sum_{i,j=1}^d \int_0^t \frac{h(X_{u-})_{ij}^2}{\operatorname{Tr}(h(X_{u-})^\top h(X_{u-}))} du = t$$

and therefore β^h is a Brownian motion on $[0,T)$.

Finally by the very definition of β^h, we have

$$\operatorname{Tr}(h(X_{t-})dB_t) = \sum_{i,j=1}^d h(X_{t-})_{ij} dB_{t,ji} = \sqrt{\operatorname{Tr}(h(X_{t-})^\top h(X_{t-}))} d\beta_t^h.$$

which proves identity (4.1). □

Finally, we state the variant of Itô's formula which we later employ. It follows easily from the usual versions like [4, Theorem 3.9.1] by arguments similar to [27, Theorem 5.4] and [3, Proposition 3.4].

Lemma 4.4. *Let X be an S_d^{++}-valued semimartingale on a stochastic interval $[0,T)$ and $f : S_d^{++} \to \mathbb{R}$ a twice continuously differentiable function. If $X_{t-} \in S_d^{++}$ for all $t \in [0,T)$ and $\sum_{0<s\leq t} \|\Delta X_s\| < \infty$ for $t \in [0,T)$, then $f(X)$ is a semimartingale on $[0,T)$ and*

$$\begin{aligned} f(X_t) =& f(X_0) + \operatorname{Tr}\left(\int_0^t \nabla f(X_{s-})^T dX_s^c\right) + \frac{1}{2} \sum_{i,j,k,l=1}^d \frac{\partial^2}{\partial x_{ij} \partial x_{kl}} f(X_{s-}) d[X_{ij}, X_{kl}]_s^c \\ &+ \sum_{0<s\leq t} \left(f(X_s) - f(X_{s-})\right). \end{aligned}$$

We are now prepared to provide a proof of Theorem 2.2.

Proof of Theorem 2.2. Since all coefficients of (2.2) are locally Lipschitz and of linear growth, standard SDE theory implies again the existence of a unique càdlàg adapted non-explosive local strong solution until the first time $T_x = \inf\{t \geq 0 : X_{t-} \in \partial S_d^+ \text{ or } X_t \notin S_d^{++}\}$ when X hits the boundary or jumps out of S_d^{++} . Hence, we only have to show $T_x = \infty$.

By the choice of g and J, all jumps have to be positive semi-definite and hence the solution X cannot jump out of S_d^{++}. This implies that $T_x = \inf\{t \geq 0 : X_{t-} \in \partial S_d^+\}$.

In the following, all statements are meant to hold on the stochastic interval $[0,T_x)$. Note that by the right continuity of X_t, a.s. $T_x > 0$. Moreover, we set $T_n = \inf\{t \in \mathbb{R}_+ : d(X_t, \partial S_d^+) \leq 1/n \text{ or } \|X_t\| \geq n\}$. Then $(T_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} T_n = T_x$, hence T_x is predictable.

We define the following processes and functions according to the notation of Proposition 3.4:

$$Z_t := \det(e^{-\beta t} X_t e^{-\beta^\top t}), \quad r_t := h(Z_t), \quad h(z) := \ln(z). \tag{4.2}$$

Then $T_x = \inf\{t > 0 \,:\, r_{t-} = 0\}$.

By Lemma 4.1 (i) we obtain

$$\operatorname{Tr}(\nabla(\det(X_{t-}))dX_t^c) = \det(X_{t-})\left[2\sqrt{\operatorname{Tr}(Q^\top Q X_{t-}^{-1})}dW_t + \operatorname{Tr}((b+\Gamma(X_{t-}))X_{t-}^{-1}) + 2\operatorname{Tr}(\beta)\right]dt,$$

with some one-dimensional Brownian W on $[0, T_x)$, whose existence is guaranteed by Lemma 4.3. Furthermore, by Lemma 4.1 (ii), Lemma 4.2 and elementary calculations we have that

$$\frac{1}{2}\sum_{i,j,k,l} \frac{\partial^2}{\partial x_{ij}\partial x_{kl}} \det(X_{t-}) d[X_{ij}, X_{kl}]_t^c = \det(X_{t-})[(1-d)\operatorname{Tr}(Q^\top Q X_{t-}^{-1})]dt.$$

According to Itô's formula, we therefore obtain by summing up the two equations,

$$\begin{aligned} d(\det(X_t)) =& 2\det(X_{t-})\sqrt{\operatorname{Tr}(Q^\top Q X_{t-}^{-1})}dW_t + \det(X_t) - \det(X_{t-}) \\ &+ \det(X_{t-})[\operatorname{Tr}((b + \Gamma(X_{t-}) + (1-d)Q^\top Q)X_{t-}^{-1}) + 2\operatorname{Tr}(\beta)]dt. \end{aligned}$$

Using again Itô's formula, we have

$$\begin{aligned} d(\ln(\det(X_t))) = & 2\sqrt{\operatorname{Tr}(Q^\top Q X_{t-}^{-1})}dW_t + \ln(\det(X_t)) - \ln(\det(X_{t-})) \\ &+ \operatorname{Tr}[(b + \Gamma(X_{t-}) - (d+1)Q^\top Q)X_{t-}^{-1}]dt + 2\operatorname{Tr}(\beta)]dt. \end{aligned}$$

We recall that the matrix exponential satisfies $\det(e^K) = e^{\operatorname{Tr} K}$ for $K \in M_d$. Hence $r_t = \ln(\det(X_t)) - 2\operatorname{Tr}(\beta)t$ and we obtain that r satisfies the following SDE,

$$dr_t = 2\sqrt{\operatorname{Tr}(Q^\top Q X_{t-}^{-1})}dW_t + \underbrace{\operatorname{Tr}\big((\underbrace{b + \Gamma(X_{t-}) - (d+1)Q^\top Q}_{\in S_d^+})X_{t-}^{-1}\big)}_{\geq 0} dt + \ln(\det(X_t)) - \ln(\det(X_{t-})). \tag{4.3}$$

Set

$$\begin{aligned} M_t &= 2\int_0^t \sqrt{\operatorname{Tr}(Q^\top Q X_{s-}^{-1})}dW_s \textit{ and} \\ P_t &= \int_0^t \operatorname{Tr}\left((b + \Gamma(X_{s-}) - (d+1)Q^\top Q)X_{s-}^{-1}\right) ds + \sum_{0<s\leq t} \ln(\det(X_s)) - \ln(\det(X_{s-})). \end{aligned}$$

We infer that $(M_t^{(n)})_{t\geq 0}$ defined by $M_t^{(n)} := \int_0^t \sqrt{\operatorname{Tr}(Q^\top Q(X_{s-}^{T_n})^{-1})}dW_s$ is a continuous martingale. Obviously, $M_t = M_t^{(n)}$ on $\{t < T_n\}$ and thus M is a continuous local martingale on $[0, T_x)$. Likewise, one can show that P is non-negative and locally bounded on $[0, T_x)$, noting that $X_s - X_{s-} \succeq 0$ for all $s \in [0, T)$ and hence $\det(X_s) \geq \det(X_{s-})$ using [24, Corollary 4.3.3].

Finally, by Proposition 3.4 we have that $T_x = \infty$ a.s. □

Remark 4.5. *The transformation trick* (4.2) *which is crucial for the analysis of this paper is motivated by the observation, that what makes McKean's argument applicable is the non-negativity of the drift in equation* (4.3) *only, whereas "the sign" of the drift of the original Wishart process does not matter at all.*

Bru's method for proving her Theorem 2.1 (ii) *for Wishart diffusions is markedly different, and consists of two steps:*

- *First assume $\beta = 0$. By applying Corollary 3.5 twice, one derives that $h(\det(X))$ is a local martingale. This is proved separately for $\delta = d+1$ and $\delta > d+1$ by choosing $h = \ln$ in the first case and $h(z) = z^{d+1-\delta}$ in the second one. Therefore, the existence of a unique global strong solution on S_d^{++} is settled.*

- *We may therefore suppose that X_t is an S_d^{++}-valued solution on* $[0,\infty)$ *of*

$$dX_t = \sqrt{X_t}dB_tQ + Q^\top dB_t^T\sqrt{X_t} + \delta Q^\top Qdt, \quad X_0 = x \in S_d^{++}.$$

where $Q \in GL(d)$ and $\delta \geq d+1$. Now, Girsanov's Theorem is applied which allows to introduce a drift by changing to an equivalent probability measure. This step generalizes a one-dimensional method by Pitman and Yor, see [6, p. 748]*. The involved arguments and calculations, which are not presented in detail in* [6]*, appear rather complicated and work seemingly only in the special case given in Theorem 2.1* (ii).

To become acquainted with the relevant matrix stochastic calculus in the context of this paper, we recommend [31]*, where also technical details of* [6] *concerning strong solutions are explained in detail.*

5. Conclusion

In this paper we have extended Bru's sufficient boundary non-attainment conditions to considerably more general SDEs on S_d^{++}, which include affine diffusions with state-independent jumps of finite variation. This allowed to infer the existence of strong solutions of a large class of affine matrix valued processes. We also obtained results for jump-diffusions with non-affine semi-martingale characteristics.

For $d = 1$ and in the diffusion case, the condition of Theorem 2.2 is a well known necessary condition for non-attainment of the boundary (see the discussion of the square Bessel processes in [35]). However, for $d > 1$ it is not clear, whether the condition $b \succeq (d+1)Q^\top Q$ for the drift is a necessary non-attainability condition or not. Only in the case $\beta = 0, \Gamma = 0, g = 0, Q = I_d$ and $b = \delta I_d$ with $\delta \in (d-1, d+1)$ it is known from [17, Theorem 1.4] that the boundary is hit. On the other hand, one knows that in the case $d = 1$ pathwise uniqueness holds, hence there exists strong solutions for all $b \succeq 0$ (even in the general setting of CBI processes, see [15, Theorem 5.1]). For $d \geq 2$, the situation seems in general to be rather complicated and therefore existence of global strong solutions remains an open problem when $b \not\succeq (d+1)Q^\top Q$.

Our method is not restricted to affine processes on S_d^+. One may for instance consider SDEs of the form

$$dX_t = \sigma(X_t)dB_t + dB_t^\top \sigma(X_t) + B(X_t)dt, \quad X(0) \in S_d$$

where $\sigma : S_d^+ \to M_d$ is locally Lipschitz on S_d^{++}. Sufficient non-attainment conditions can here be obtained by choosing a positive drift $B : S_d^+ \to S_d$ in such a way, that the drift of $\ln(\det(X))$ is non-negative and therefore makes Proposition 3.4 applicable (cf. proof of Theorem 2.2). Finally, we remark that the method can be generalized to state-spaces D other than S_d^+, as long as the existence of an appropriate function $g : D \to \mathbb{R}_+$ is guaranteed, such that $g^{-1}(0) = \partial D$. For instance, similar (but simpler) arguments as the ones of the proof of Theorem 2.2 yield a rigorous proof of the non-attainment condition formulated in [9, Section 6] for affine jump-diffusions on canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$. Here one takes $g = x_1 \cdot x_2 \cdot \dots \cdot x_m$. This topic might be interesting for further investigation.

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