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Constructing the transition laws of affine processes: A simplified point of view

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CONSTRUCTING THE TRANSITION LAWS OF AFFINE PROCESSES: A SIMPLIFIED POINT OF VIEW

A DIDACTIC NOTE

EBERHARD MAYERHOFER

Abstract. In this note we revisit the construction of the transition laws of affine processes, by modifying slightly the original arguments of Duffie, Filipović and Schachermayer (2003). We replace part of their abstract construction by a simplified version of the central limit theorem, and reduce the multi-dimensional situation to a one-dimensional one (i.e. for CBI processes on $\mathbb{R}_+$), by block-diagonalization of the diffusion coefficients.

Contents

Let $X = (X_t^x)_{t\geq 0, x\in D}$ be a stochastically continuous Markov process with state-space $D = \mathbb{R}_+^m \times \mathbb{R}^n$. We say X is an affine process, if its characteristic function is exponentially affine in the state-variable, that is,

$$\mathbb{E}^x[e^{\langle u, X_t\rangle}] = e^{\phi(t,u)+\langle \psi(t,u), x\rangle}$$

holds for all $u \in i\mathbb{R}^d$. It is known, that in that case $\phi(t,u)$ and $\psi(t,u)$ can be continuously extended to $\mathbb{C}_-^m \times i\mathbb{R}^d$, and there exist functions F and R such that

$$\partial_t \phi(t,u)|_{t=0} = F(u), \quad \partial_t \psi(t,u)|_{t=0} = R(u)$$

Moreover, $\phi(t,u)$ and $\psi(t,u)$ are differentiable and satisfy the so-called generalized Riccati differential equations

$$\dot{\phi}(t,u) = F(\psi(t,u), \quad \phi(0,u) = 0 \tag{1}$$

$$\dot{\psi}(t,u) = R(\psi(t,u), \quad \psi(0,u) = u. \tag{2}$$

The functions F and R, which are in Lévy-Khintchine form on $\mathbb{R}^d$, can be be described explicitly in terms of their corresponding Lévy triplets (in the affine Literature the so-called admissible parameter set). For more detailed information, see [1, 5].

In this note we are interested in understanding better the existence proof of [1, chapter 7]*, which proceeds as follows: given F, R, we try to solve equations* (1)–(2) *and show that $e^{\phi(t,u)+\langle\psi(t,u),x\rangle}$ is the Fourier-Laplace transform of a sub-stochastic measure $p(t,x,d\xi)$ on D, for each $t \geq 0$, $x \in D$. Having that, yields existence of* *an affine Markov process, because $p(t,x,d\xi)$ are knitted together by the Chapman-Kolmogorov equation, which are implied by equations* (1)–(2).

This paper contributes to a simplification of the basics of affine processes. For those readers, who have an elementary knowledge of Lévy processes, I recommend to check first Martin Keller-Ressel's simplified approach [4, part 1, section 2] of deriving necessary parametric restrictions concerning F and R, and then try to understand chapter 7 of [1] by means of these notes.

In section 1, I consider the special case of CBI-processes on the positive real line $\mathbb{R}_+ = [0,\infty)$. Section 2 explains how the multi-dimensional situation $D = \mathbb{R}_+^m \times \mathbb{R}$ can be reduced to the one-dimensional one.

1. The core: approximation of function R for CBI processes

In this section we consider the one-dimensional case, that is $D = \mathbb{R}_+$ ($m = 1, n = 0$ in the notation above). We recall notation and define,

Definition 1.

- Let χ be some truncation function. In the following R is a function of the form
$$R(u) = \alpha u^2 + \beta u - c + \int_{\mathbb{R}_{++}} \left(e^{u\xi} - 1 - u\chi(\xi)\right)\mu(d\xi)$$
where α, $c \geq 0$, $\beta \in \mathbb{R}$, and μ is a non-negative σ-finite measure supported on $\mathbb{R}_{++}$ which integrates $\xi^2 \wedge 1$. We recall that R is well defined on $\mathbb{C}_-$ and is analytic on $\mathbb{C}_{--}$.
- $\mathcal{C}$ is defined as the set of functions which admit a Lévy-Khintchine representation of substochastic ID measures on $\mathbb{R}_+$. That is,
$$\mathcal{C} := \{f : \mathbb{C}_- \to \mathbb{C} \mid f(u) = \beta u - c + \int_{\mathbb{R}_{++}} \left(e^{u\xi} - 1\right)\mu(d\xi)\}$$
where $c \geq 0$, $\beta \in \mathbb{R}_+$ and μ is a non-negative σ-finite measure supported on $\mathbb{R}_{++}$ which integrates $\xi \wedge 1$.
- In addition, we introduce a set $\mathcal{E}$ which is defined as the set of functions which admit a Lévy-Khintchine representation of substochastic ID measures on $\mathbb{R}$, and are of the form $\mathcal{C}$, apart from the linear drift component (which may be of negative sign). That is,
$$\mathcal{E} := \{f : \mathbb{C}_- \to \mathbb{C} \mid f(u) = \beta u - c + \int_{\mathbb{R}_{++}} \left(e^{u\xi} - 1\right)\mu(d\xi)\}$$
where $c \geq 0$, $\beta \in \mathbb{R}$ and μ is a non-negative σ-finite measure supported on $\mathbb{R}_{++}$ which integrates $\xi \wedge 1$.

Lemma 2. *On $\mathbb{C}_-$, $R(u)$ can be locally uniformly approximated by a sequence $(R_n)_n$, where $R_n \in \mathcal{E}$ for each $n \geq 1$.*

Proof. We show first that for the sequence $\nu^{(n)}(d\xi) := \mu(d\xi)1_{\|\xi\| \geq 1/n}$ we have
$$\int_{(0,1)} \left(e^{u\xi} - 1 - u\xi\right)\nu^{(n)}(d\xi) \to \int_{(0,1)} \left(e^{u\xi} - 1 - u\xi\right)\mu(d\xi)$$
locally uniformly, as $n \to \infty$. Indeed, using the identity
$$e^{u\xi} - 1 - u\xi = (u\xi)^2 \int_0^1 (1-t)e^{tu\xi}dt,$$

we obtain, for $\|u\| \leq R$

$$\int_{(0,1/n)} \left(e^{u\xi} - 1 - u\xi\right) \mu(d\xi) = \int_{(0,1/n)} (u\xi)^2 \int_0^1 (1-t)e^{tu\xi} dt\mu(d\xi) \to 0$$
$$\leq R^2 e^{R/n} \int_{(0,1/n)} \xi^2 \mu(d\xi) \to 0$$

as $n \to \infty$, because $\xi^2 \wedge 1\mu(d\xi)$ extends to a bounded measure on $\mathbb{R}_+$, by definition.

Next we turn to the approximation of αu^2. Without loss of generality, we may assume $\alpha = \frac{1}{2}$. We see that we need to approximate the cumulant generating function of a standard Gaussian on $\mathbb{C}_-$, by elements of $\mathcal{E}$. Let $(Z_k)_k$ be a sequence of i.i.d. Poisson random variables with parameter 1. Then by a stripped down version of the central limit theorem, we have with $S_n := \sum_{k=1}^n Z_k$ for the sequence $Y_n := \frac{S_n - n}{\sqrt{n}}$ that

$$\mathbb{E}[e^{uY_n}] \to e^{\frac{u^2}{2}}, \quad as \quad n \to \infty.$$

By the additivity of the Poisson distribution we have that $S_n \sim P(n)$. Therefore for $u \in \mathbb{C}$ we have

$$\mathbb{E}[e^{uY_n}] = e^{-\sqrt{n}u}\mathbb{E}[e^{\frac{u}{\sqrt{n}}S_n}] = e^{-\sqrt{n}u}\exp(n(e^{\frac{u}{\sqrt{n}}} - 1))$$

with the dirac measure δ_d at $\xi = d$, we therefore obtain that

$$\mathcal{E} \ni \log \mathbb{E}[e^{uY_n}] = -\sqrt{n}u + \int (e^{u\xi} - 1)n\delta_{\frac{1}{\sqrt{n}}}(d\xi) \to \frac{u^2}{2}$$

locally uniformly, as $n \to \infty$.

Altogether, we have shown that $R^{(n)} \to R$ locally uniformly as $n \to \infty$, where $R^{(n)}$ is defined by the admissible quadruplet $(\alpha^{(n)}, \beta^{(n)}, \gamma^{(n)}, \mu^{(n)})$ given by

$$\alpha^{(n)} := 0, \quad \beta^{(n)} := \beta - \int_{(0,1]} \xi \, \nu_n(d\xi) - \sqrt{n}, \quad \gamma^{(n)} := \gamma, \quad \mu^{(n)} := \nu^{(n)} + n\delta_{\frac{1}{\sqrt{n}}}$$

and by construction $R_n \in \mathcal{E}$, for each $n \geq 1$.

This was the intuitive counterpart of the approximation in [1, Proof of Lemma 7.5]. Clearly there is no need to use Poisson random variables for this approximation, but only ID variables. Note further, that since the FLT of an ID random variable has no zeros, one can take the logarithm of the approximating FLTs and obtain the asserted approximation result. □

Consider now the generalized Riccati equation (1)–(2). We note that always $F \in \mathcal{C}$. Using Picard iteration, one can show that for the case that $R \in \mathcal{E}$, we have that $\psi(t, \cdot)$ and $\phi(t, \cdot)$ lie in $\mathcal{C}$, for each $t \geq 0$. Therefore, for general R it suffices to know that by Lemma 2 R can be approximated locally uniformly by elements R_k in $\mathcal{E}$, since then for each t, $\psi^{(k)}(t, u) \to \psi(t, u)$ (and similarly for ϕ) locally uniformly as $k \to \infty$, and therefore $\phi(t, u) + \psi(t, \cdot) \cdot x \in \mathcal{C}$, i.e., is the cumulant generating function of a substochastic measure on D. For the well elaborated details we refer to [1, chapter 7].

2. The general case: $D = \mathbb{R}_+^m \times \mathbb{R}^n$

In general F and R are of Lévy-Khintchine form on $\mathbb{R}^d$, and their Lévy triplets satisfy a pack of –maybe at first sight complicated– parameter restrictions, see [1, Definition 2.6 and Theorem 2.7]. I avoid to recapitulate those conditions in detail here.

We observe first, that the diffusion coefficients α_i $i = 1, \dots, m$ may be chosen in block-diagonal form. In fact, we have by [2, Lemma 7.1],

Lemma 3. *Let Y be an affine process on D with diffusion coefficients $\overline{\alpha}_i$, $i = 1, \dots, m$. Then there exists a linear automorphism Λ of D such that $X := \Lambda Y$ is an affine process with block diagonal diffusion, that is $\alpha_{i,IJ} = \alpha_{i,JI} = 0$, for $i = 1, \dots, m$.*

Therefore, once existence for X is shown, we may obtain a process Y with non-zero off-block-diagonal diffusion coefficients by a linear transformation of the state-space. For more details concerning the transformation of R we refer to the proof of [2, Lemma B.5]. In the following, we therefore assume without loss of generality block-diagonal diffusion.

Let χ be any truncation function on $\mathbb{R}^d$.[1] Using the simplification above, for $i = 1, \dots, m$ we can write R_i as follows

$$R_i(u) = \alpha_{i,ii} u_i^2 + \beta_i^\top u + \int_{D \setminus 0} e^{\langle u, \xi \rangle} - 1 - \langle u, \chi(\xi) \rangle \mu_i d\xi$$

where $\alpha_{i,ii} \geq 0$ $(\beta_i)_j \geq 0$ for $j \neq i$, $j = 1, \dots, m$, and μ_i is a σ-finite Borel measure on $D \setminus \{0\}$ which integrates $\|\xi\|^2 \wedge 1$ and $(\xi_j \wedge 1)$ where $j \neq i$, $j = 1, \dots, m$. Approximating R in Lévy-Khintchine form on D modulo linear drift now means that we have to approximate R by elements of $\mathcal{C}$ modulo linear drift[2] as defined in [1, equation (7.1)]. This can be obtained by performing the following two steps, similarly as in the proof of Lemma 2:

- Approximate the measures μ_i by $\nu_i^{(n)}(d\xi) := 1_{\|\xi\| > 1/n} \mu_i(d\xi)$, each n.
- For each $i = 1, \dots, m$, approximate $\alpha_{i,ii} u_i^2$ by means of cumulant generating functions of Poisson distributions. Define $\nu_i^{(n)}(d\xi) := n\alpha_{i,ii}\, \delta_{\frac{e_i}{\sqrt{n}}}(d\xi)$, where e_i is the i-ths canonical basis vector. $R_i^{(n)}$ is accordingly defined by the admissible quadruplet $(\alpha_i^{(n)} = 0, \beta_i^{(n)}, \gamma_i^{(n)} = \gamma_i, \mu_i^{(n)})$, where

$$\beta_i^{(n)} := \beta_i - \int_{0 < \|\xi\| < 1} \xi \, \nu_i^{(n)}(d\xi) - \alpha_{i,ii} \sqrt{n} e_i, \quad \mu_i^{(n)} := \nu_i^{(n)} + n\alpha_{i,ii} \delta_{\frac{e_i}{\sqrt{n}}}(d\xi)$$

Finally, using Picard iteration, we may obtain that $\phi(t, \cdot) + \langle \psi(t, \cdot), x \rangle$ is of (sub-stochastic) Lévy-Khintchine form on D, and the rest is history.

3. Concluding remarks

The possibility to approximate a diffusion with jump processes of increasing finite variation – which is the intuition behind the presented proof variant – is well known, and used in finance (one may think of Brownian motion approximated by compensated Poisson processes). In the affine framework, this has also been mentioned by [3, Example 2.12], in a slightly different context. Nevertheless, the existence proof of [3] is elaborated still in the somehow abstract manner of [1], and therefore I found it valuable to present this simplifying variant.

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[1] Unlike in [1], the truncation function is (and need) not (be) specified.

[2] the linear drift will in general be not in D

This note aims at explaining a well established result of [1], and therefore is not planned for publication. I appreciate comments and suggestions for improvements by email.

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