VIF Vienna Institute of Finance

Working Paper Series

Working Paper No. 23

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Preceding version: September 2009, authored by E.M. and A.G.S title: “A characterization of conservative affine Markov processes”

Current version: October 2009

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A CHARACTERIZATION OF THE MARTINGALE PROPERTY OF EXPONENTIALLY AFFINE PROCESSES

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Abstract. We consider local martingales of exponential form $M = e^X$ or $\mathscr{E}(X)$ where X denotes one component of a multivariate affine process in the sense of Duffie, Filipović and Schachermayer [8]. By completing the characterization of conservative affine processes in [8, Chapter 9], we provide deterministic necessary and sufficient conditions in terms of the parameters of X for M to be a true martingale.

1. Introduction

A classical question in probability theory comprises the following. Suppose the ordinary resp. stochastic exponential $M = \exp(X)$ resp. $\mathscr{E}(X)$ of some process X is a positive *local martingale* and hence a supermartingale. Then under what (if any) additional assumptions is it in fact a *true martingale*?

This seemingly technical question is of considerable interest in diverse applications, for example absolute continuity of distributions of stochastic processes (cf. e.g. [3] and the references therein), absence of arbitrage in financial models (cf. e.g. [6]) or verification of optimality in stochastic control (cf. e.g. [9]).

In a general semimartingale setting it has been shown in [11] that any supermartingale M is a martingale if and only if it is non-explosive under the associated *Föllmer measure*. However, this general result is hard to apply in concrete models, since it is expressed in purely probabilistic terms. Consequently, there has been extensive research focused on exploiting the link between martingales and non-explosion in various more specific settings, see e.g. [24]. In particular, *deterministic* necessary and sufficient conditions for the martingale property of M have been obtained if X is a one-dimensional diffusion (cf. e.g. [7, 2] and the references therein).

For processes with jumps, the literature is more limited and mostly focused on sufficient criteria as in e.g. [20, 16, 21, 15]. By the independence of increments and the Lévy-Khintchine formula, no extra assumptions are needed for M to be a true martingale if X is a Lévy process. For the more general class of *affine processes*

2000 *Mathematics Subject Classification.* 60G44, 60J25, 60J75.

Key words and phrases. Exponential martingales, affine processes, semimartingale characteristics, conservative processes.

E.M. and A.G.S gratefully acknowledge financial support from WWTF (Vienna Science and Technology Fund). J. M.-K. gratefully acknowledges support from the FWF (Austrian Science Fund) under grant P19456.

characterized in [8] the situation becomes more involved. While no additional conditions are needed for continuous affine processes, this no longer remains true in the presence of jumps (cf. [15, Example 3.9]). In this situation a necessary and sufficient condition for one-factor models has been established in [18, Theorem 2.5], whereas easy-to-check sufficient conditions for the general case are provided in [15, Theorem 3.1].

In the present study, we complement these results by sharpening [15, Theorem 3.1] in order to provide deterministic necessary and sufficient conditions for the martingale property of $M = \mathscr{E}(X^i)$ resp. $\exp(X^i)$ in the case where X^i is one component of a general non-explosive affine process X. As in [18, 15] these conditions are expressed in terms of the admissible *parameters* which characterize the distribution of X (cf. [8]).

Since we also use the linkage to non-explosion, we first complete the characterization of *conservative*, i.e. non-explosive, affine processes from [8, Chapter 9]. Generalizing the arguments from [15], we then establish that M is a true martingale if and only if it is a local martingale and a related affine process is conservative. Combined with the characterization of local martingales in terms of semimartingale characteristics [14, Lemma 3.1] this then yields necessary and sufficient conditions for the martingale property of M.

The article is organized as follows. In Section 2, we recall terminology and results on affine Markov processes from [8]. Afterwards, we characterize conservative affine processes. Subsequently, in Section 4, this characterization is used to provide necessary and sufficient conditions for the martingale property of exponentially affine processes. Appendix A develops ODE comparison results in a general non-Lipschitz setting that are used to establish the results in Section 3, whereas Appendix B contains a simple consequence of stochastic continuity that is used repeatedly in the proofs of Section 4.

2. Affine processes

For stochastic background and terminology, we refer to [13, 22]. We work in the setup of [8], that is we consider a time-homogeneous Markov process with state space $D := \mathbb{R}^m_+ \times \mathbb{R}^n$, where $m, n \geq 0$ and $d = m + n \geq 1$. We write $p_t(x, d\xi)$ for its transition function and let $(X, \mathbb{P}_x)_{x\in D}$ denote its realization on the canonical filtered space $(\Omega, \mathscr{F}^0, (\mathscr{F}^0_t)_{t\in\mathbb{R}_+})$ of paths $\omega : \mathbb{R}_+ \to D_\Delta$ (the one-point-compactification of D). For every $x \in D$, $\mathbb{P}_x$ is a probability measure on $(\Omega, \mathscr{F}^0)$ such that $\mathbb{P}_x(X_0 = x) = 1$ and the Markov property holds, i.e.

$$\begin{aligned}\mathbb{E}_x(f(X_{t+s})|\mathscr{F}^0_s) &= \int_D f(\xi)p_s(X_t, d\xi)\\ &= \mathbb{E}_{X_t}(f(X_s)), \quad \mathbb{P}_x - a.s. \quad \forall t, s, \in \mathbb{R}_+,\end{aligned}$$

for all bounded Borel-measurable functions $f : D \to \mathbb{C}$. The Markov process $(X, \mathbb{P}_x)_{x\in D}$ is called *conservative* if $p_t(x, D) = 1$, *stochastically continuous* if we have $p_s(x, \cdot) \to p_t(x, \cdot)$ weakly on D, for $s \to t$, for every $(t, x) \in \mathbb{R}_+ \times D$, and *affine* if, for every $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$, the characteristic function of $p_t(x, \cdot)$ is of the form

$$\int_D e^{\langle u,\xi\rangle} p_t(x, d\xi) = \exp\left(\psi_0(t, u) + \langle \psi(t, u), x\rangle\right), \quad \forall x \in D, \tag{2.1}$$

for some $\psi_0(t, u) \in \mathbb{C}$ and $\psi(t, u) = (\psi_1(t, u), \ldots, \psi_d(t, u)) \in \mathbb{C}^d$. For every stochastically continuous affine process, the mappings $(t, u) \mapsto \psi_0(t, u)$ and $(t, u) \mapsto \psi(t, u)$ can be characterized in terms of the following quantities:

Definition 2.1. *Denote by $h = (h_1, \ldots, h_d)$ the truncation function on $\mathbb{R}^d$ defined by*

$$h_k(\xi) := \begin{cases} 0, & \text{if } \xi_k = 0, \\ (1 \wedge |\xi_k|)\frac{\xi_k}{|\xi_k|}, & \text{otherwise.} \end{cases}$$

Parameters $(\alpha, \beta, \gamma, \kappa)$ are called admissible*, if*

- $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$ *with symmetric positive semi-definite $d \times d$-matrices α_j such that $\alpha_j = 0$ for $j \geq m + 1$ and $\alpha_j^{kl} = 0$ for $0 \leq j \leq m$, $1 \leq k, l \leq m$ unless $k = l = j$;*
- $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_d)$ *where κ_j is a Borel measure on $D\backslash\{0\}$ such that $\kappa_j = 0$ for $j \geq m + 1$ as well as $\int_{D\backslash\{0\}} \|h(\xi)\|^2 \kappa_j(d\xi) < \infty$ for $0 \leq j \leq m$ and*

$$\int_{D\backslash\{0\}} h_k(\xi)\kappa_j(d\xi) < \infty, \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j;$$

- $\beta = (\beta_0, \beta_1, \ldots, \beta_d)$ *with $\beta_j \in \mathbb{R}^d$ such that $\beta_j^k = 0$ for $j \geq m+1$, $1 \leq k \leq m$ and*

$$\beta_j^k + \int_{D\backslash\{0\}} h_k(\xi)\kappa_j(d\xi) \geq 0, \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j.$$

- $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_d)$*, where $\gamma_j \in \mathbb{R}_+$ and $\gamma_j = 0$ for $j = m + 1, \ldots, d$.*

Affine Markov processes and admissible parameters are related as follows (cf. [8, Theorem 2.7] and [19, Theorem 5.1]):

Theorem 2.2. *Let $(X, \mathbb{P}_x)_{x\in D}$ be a stochastically continuous affine process. Then there exist admissible parameters $(\alpha, \beta, \gamma, \kappa)$ such that $\psi_0(t, u)$ and $\psi(t, u)$ are given as solutions to the generalized Riccati equations*

$$\partial_t \psi(t, u) = R(\psi(t, u)), \qquad \psi(0, u) = u, \tag{2.2}$$

$$\partial_t \psi_0(t, u) = R_0(\psi(t, u)), \quad \psi_0(0, u) = 0, \tag{2.3}$$

where $R = (R_1, \ldots, R_d)$ and for $0 \leq i \leq d$,

$$R_i(u) := \frac{1}{2}\langle \alpha_i u, u\rangle + \langle \beta_i, u\rangle - \gamma_i + \int_{D\backslash\{0\}} \left(e^{\langle u,\xi\rangle} - 1 - \langle u, h(\xi)\rangle\right)\kappa_i(d\xi). \tag{2.4}$$

Conversely, for any set $(\alpha, \beta, \gamma, \kappa)$ of admissible parameters there exists a unique stochastically continuous affine process such that (2.1) *holds for all $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$ where ψ_0 and ψ are given by* (2.3) *and* (2.2)*.*

Since any stochastically continuous affine process $(X, \mathbb{P}_x)_{x\in D}$ is a Feller process (cf. [8, Theorem 2.7]), it admits a càdlàg modification and hence can be realized on the space of càdlàg paths $\omega : \mathbb{R}_+ \to D_\Delta$. If $(X, \mathbb{P}_x)_{x\in D}$ is also conservative it actually turns out to be a semimartingale in the usual sense and hence can be realized on the *Skorokhod space* $(\mathbb{D}^d, \mathscr{D}^d, (\mathscr{D}^d_t)_{t\in\mathbb{R}_+})$ of D- rather than D_Δ-valued càdlàg paths endowed with its natural filtration (cf. [13, Chapter VI]). In this case, the semimartingale characteristics of $(X, \mathbb{P}_x)_{x\in D}$ are given in terms of the admissible parameters:

Theorem 2.3. *Let $(X, \mathbb{P}_x)_{x\in D}$ be a conservative, stochastically continuous affine process and $(\alpha, \beta, \gamma, \kappa)$ the related admissible parameters. Then $\gamma = 0$ and for any $x \in D$, X is a semimartingale on $(\mathbb{D}^d, \mathscr{D}^d, (\mathscr{D}^d_t)_{t\in\mathbb{R}_+}, \mathbb{P}_x)$ with characteristics (B, C, ν) given by*

$$B_t = \int_0^t \left(\beta_0 + \sum_{j=1}^d \beta_j X^j_{s-}\right) ds, \tag{2.5}$$

$$C_t = \int_0^t \left(\alpha_0 + \sum_{j=1}^d \alpha_j X^j_{s-}\right) ds, \tag{2.6}$$

$$\nu(dt, d\xi) = \left(\kappa_0(d\xi) + \sum_{j=1}^d X^j_{s-}\kappa_j(d\xi)\right) dt, \tag{2.7}$$

relative to the truncation function h. Conversely, let X' be a D-valued semimartingale defined on some filtered probability space $(\Omega', \mathscr{F}', (\mathscr{F}'_t), \mathbb{P}')$. If $\mathbb{P}'(X'_0 = x) = 1$ and X' admits characteristics of the form (2.5)-(2.7) *with X_- replaced by X'_-, then $\mathbb{P}' \circ X'^{-1} = \mathbb{P}_x$.*

Proof. $\gamma = 0$ is shown in [8, Proposition 9.1]. The remaining assertions follow from [8, Theorem 2.12], since Stricker's theorem and [12, Proposition 1.1] show that X is a semimartingale with characteristics (B, C, ν) relative to the uncompleted filtration $(\mathscr{D}^d_t)_{t\in\mathbb{R}_+}$. □

Note that X is a Markov process relative to the filtration $(\mathscr{D}^d_t)_{t\in\mathbb{R}_+}$ by [22, Proposition 2.14].

3. Conservative affine processes

In view of Theorem 2.3, the powerful toolbox of semimartingale calculus is made available, provided that the Markov process $(X, \mathbb{P}_x)_{x\in D}$ under consideration is conservative. Hence is desirable to characterize this property in terms of the parameters of X. This is done in the present section. The main statement is Theorem 3.3, which completes the discussion of conservativeness in [8, Chapter 9].

Let us first introduce some definitions and notation. The partial order on $\mathbb{R}^m$ induced by the natural cone $\mathbb{R}^m_+$ is denoted by $\leqq$. That is, $x \leqq 0$ if and only if $x_i \leq 0$ for $i = 1, \ldots, m$. A function $g : D_g \to \mathbb{R}^m$ is *quasimonotone increasing* on

$D_g \subset \mathbb{R}^m$ (*qmi* in short, for a general definition see section A) if and only if for all $x, y \in D_g$ and $i = 1, \dots, m$ the following implication holds true:

$$(x \leqq y, \quad x_i = y_i) \;\Rightarrow\; g_i(x) \le g_i(y).$$

In the sequel we write $\mathbb{R}_{--} := (-\infty, 0)$ and $\mathbb{C}_{--} := \{c \in \mathbb{C} \mid \mathrm{Re}(c) \in \mathbb{R}_{--}\}$. Moreover, we introduce the index set $\mathcal{I} := \{1, \dots, m\}$ and, accordingly, define by $u_{\mathcal{I}}$ the projection of the d–dimensional vector u onto the first m coordinates.

For the statements in this section, we rely on abstract properties of the involved generalized Riccati differential equations as summarized in the following

Lemma 3.1. *The affine transform formula* (2.1) *also holds for* $u = (u_{\mathcal{I}}, 0) \in \mathbb{R}^d_-$. *More specifically, there exist jointly continuous mappings* $\psi_0(t, (u_{\mathcal{I}}, 0)) : \mathbb{R}_+ \times \mathbb{R}^m_- \to \mathbb{R}_-$ *and* $\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0)) : \mathbb{R}_+ \times \mathbb{R}^m_- \to \mathbb{R}^m_-$ *satisfying*

$$\partial_t \psi_0(t, (u_{\mathcal{I}}, 0)) = R_0(\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))), \qquad \psi_0(0, (u_{\mathcal{I}}, 0)) = 0, \tag{3.1}$$

$$\partial_t \psi_{\mathcal{I}}(t, (v, 0)) = R_{\mathcal{I}}((\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0)), 0), \quad \psi_{\mathcal{I}}(0, (u_{\mathcal{I}}, 0)) = u_{\mathcal{I}}, \tag{3.2}$$

where $R_0, R_{\mathcal{I}}$ *are continuous functions on* $\mathbb{R}^m_-$ *such that* $R_0(0) \le 0$, $R_{\mathcal{I}}(0) \leqq 0$ *and where* $R_{\mathcal{I}}((u_{\mathcal{I}}, 0))$ *is locally Lipschitz on* $\mathbb{R}^m_{--}$ *and qmi on* $\mathbb{R}^m_-$.

Moreover, $\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))$ *restricts to a* $\mathbb{R}^m_{--}$*-valued global solution* $\psi^\circ_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))$ *of* (3.2) *on* $\mathbb{R}_+ \times \mathbb{R}^m_{--}$.

Proof. By [19] a *stochastically continuous* affine processes is *regular* in the sense of [8]. Hence, the first statement is a consequence of [8, Proposition 6.4]. The regularity of R_0 and $R_{\mathcal{I}}$ follows from [8, Lemma 5.3 (i) and (ii)]. Equation (2.4) shows $R_0(0) \le 0$ and $R_{\mathcal{I}}(0) \leqq 0$. The map $v \mapsto R_{\mathcal{I}}((v, 0))$ is qmi on $\mathbb{R}^m_-$ by [17, Lemma 4.6], whereas the last assertion is stated in [8, Proposition 6.1]. □

In a first step, we establish the minimality of $\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))$ among all solutions of (3.2) with respect to the partial order $\leqq$.

Proposition 3.2. *Let* $T > 0$ *and* $u_{\mathcal{I}} \in \mathbb{R}^m_-$. *If* $g(t) : [0, T) \to \mathbb{R}^m_-$ *is a solution of*

$$\partial_t g(t) = R_{\mathcal{I}}(g(t), 0), \quad g(0) = u_{\mathcal{I}}, \tag{3.3}$$

then $g(t) \geqq \psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))$, *for all* $t < T$.

Proof. The properties of $R_{\mathcal{I}}$ established in Lemma 3.1 allow this conclusion by a use of Corollary A.3: For an application of the latter, we make the obvious choices $f = R_{\mathcal{I}}$, $D_f = \mathbb{R}^m_-$. Then we know that for $u^\circ_{\mathcal{I}} \in \mathbb{R}^m_{--}$ we have $g(t) \ge \psi^\circ_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))$, for all $t < T$. Now letting $u^\circ_{\mathcal{I}} \to u_{\mathcal{I}}$ and using the continuity of $\psi_{\mathcal{I}}$ as asserted in Lemma 3.1 yields the assertion. □

We now state the main result of this section. The implication (i) ⇒ (ii) completes the characterization of conservative affine processes in [8, Proposition 9.1].

Theorem 3.3. *The following statements are equivalent:*

(i) $(X, \mathbb{P}_x)_{x \in D}$ *is conservative,*
(ii) $R_0(0) = 0$ *and there exists no non-trivial* $\mathbb{R}^m_-$*-valued local solution* $g(t)$ *of* (3.3) *with* $g(0) = 0$.

Moreover, each of these statements implies that $R(0) = 0$.

Proof. (i)⇒(ii): By definition, X is conservative if and only if for all $t \geq 0$ and $x \in D$ we have

$$1 = p_t(x, D) = e^{\psi_0(t,0)+\langle \psi(t,0),x\rangle} = e^{\psi_0(t,0)+\langle \psi_{\mathcal{I}}(t,0),x_{\mathcal{I}}\rangle},$$

because $\psi_i(t, (u_{\mathcal{I}}, 0)) = 0$, for $i = m + 1, \ldots, d$. This in turn is equivalent to

$$\psi_0(t, 0) = 0 \text{ and } \psi_{\mathcal{I}}(t, 0) = 0 \quad \forall t \geq 0. \tag{3.4}$$

Now assume, by contradiction, that for some $T > 0$ there exists a $\mathbb{R}^m_-$-valued solution $g \neq 0$ of (3.3) on $[0, T)$ such that $g(0) = 0$. By Proposition 3.2, we have $\psi_{\mathcal{I}}(t, 0) \leqq g(t)$ for $t < T$, hence for some $t_0 \in (0, T)$ and some $i \in \mathcal{I}$, $\psi_i(t_0, 0) \leq g_i(t_0) < 0$, but this violates (3.4). Or assume that $R_0(0) \neq 0$. This implies $R_0(0) < 0$. But then for all $t > 0$, $\psi_0(t, 0) \leq R_0(0)t < 0$, which again violates (3.4). Hence statement (ii) holds.

(ii) ⇒(i): Assume, by contradiction, that for some $t_0 > 0$ and $x \in D$, $p_{t_0}(x, D) \neq 1$. Then (3.4) implies that either $\psi_{\mathcal{I}}(t_0, 0) \neq 0$ or $\psi_0(t_0, 0) \neq 0$. In the first case, Lemma 3.1 yields that $\psi_{\mathcal{I}}(t, 0)$ is a non-trivial $\mathbb{R}^m_-$-valued solution of (3.3) with initial data $g(0) = 0$, which contradicts the second part of statement (ii). Assume now, by contradiction, $\psi_0(t_0, 0) < 0$, but $\psi_{\mathcal{I}}(\cdot, 0) = 0$ on $\mathbb{R}_+$. Then we obtain $R_0(0)t_0 = \psi_0(t_0, 0) < 0$, which contradicts $R_0(0) = 0$.

Finally, we show that each of the equivalent statements implies $R(0) = 0$. Suppose, by contradiction, that $R(0) \neq 0$. Since $R_i(0) = 0$ for $i = m + 1, \ldots, d$, there exist $\varepsilon > 0$ and $i \in \mathcal{I}$ such that $R_i(0) \leq -\varepsilon < 0$. Let now $u_n^\circ \in \mathbb{R}^m_{--}$ such that $u_n^\circ \to 0$ as $n \to \infty$. In view of the continuity of R and $\psi_{\mathcal{I}}$, dominated convergence yields

$$\psi_i(t, 0) = \lim_{n\to\infty} \psi_i(t, (u_n^\circ, 0)) = \lim_{n\to\infty} \int_0^t R_i(\psi_{\mathcal{I}}(s, (u_n^\circ, 0)))ds = R_i(0)t \leq -\varepsilon t,$$

which is impossible due to (3.4). □

Remark 3.4. (i) By Definition 2.1, $R_0(0) = 0$, $R(0) = 0$ is equivalent to $\gamma = 0$. This means that the infinitesimal generator of the associated Markovian semi-group has zero potential, see [8, Equation (2.12)]. If an affine process with $\gamma = 0$ fails to be conservative, then it must have state-dependent jumps.

(ii) The comparison results established in Appendix A are the major tool for proving Proposition 3.2. They are quite general and therefore allow for a similar characterization of conservativeness of affine processes on geometrically more involved state-spaces (as long as they are proper closed convex cones). In particular, such a characterization can be derived for affine processes on the cone of symmetric positive semidefinite matrices of arbitrary dimension, see also [5, Remark 2.5].

(iii) Conservativeness of $(X, \mathbb{P}_x)_{x\in D}$ and uniqueness for solutions of the ODE (3.3) can be ensured by requiring

$$\int_{D\setminus\{0\}} \left(|\xi_k| \wedge |\xi_k|^2\right) \kappa_j(d\xi) < \infty, \quad 1 \leq k, j \leq m, \tag{3.5}$$

as in [8, Lemma 9.2], which implies that $R_{\mathcal{I}}(\cdot, 0)$ is locally Lipschitz continuous on $\mathbb{R}^m_-$.

(iv) If $m = 1$, conservativeness corresponds to uniqueness of a one dimensional ODE and can be characterized more explicitly: [8, Corollary 2.9], [10, Theorem 4.11] and Theorem 3.3 yield that $(X, \mathbb{P}_x)_{x \in D}$ is conservative if and only if either (3.5) holds or

$$\int_{0-} \frac{1}{R_1(u_1, 0)} du_1 = -\infty, \tag{3.6}$$

where $\int_{0-}$ denotes an integral over an arbitrarily small left neighborhood of 0.

4. Exponentially affine martingales

We now turn to the characterization of exponentially affine martingales. Henceforth, let $(X, \mathbb{P}_x)_{x \in D}$ be the canonical realization on $(\mathbb{D}^d, \mathscr{D}^d, (\mathscr{D}^d_t)_{t \in \mathbb{R}_+}$ of a conservative, stochastically continuous affine process with corresponding admissible parameters $(\alpha, \beta, 0, \kappa)$. Our first lemma shows that the local martingale property of stochastic exponentials of components of X can be read directly from the corresponding parameters.

Lemma 4.1. *Let $i \in \{1, \ldots, d\}$. Then $\mathscr{E}(X^i)$ is a local $\mathbb{P}_x$-martingale for all $x \in D$ if and only if*

$$\int_{\{|\xi_i|>1\}} |\xi_i| \kappa_j(d\xi) < \infty, \quad 0 \leq j \leq d, \tag{4.1}$$

and

$$\beta^i_j + \int_{D \setminus \{0\}} (\xi_i - h_i(\xi)) \kappa_j(d\xi) = 0, \quad 0 \leq j \leq d. \tag{4.2}$$

Proof. $\Leftarrow$: Fix $x \in D$. The predictable, locally bounded process X_- is almost surely bounded on $[0, T]$. Hence it follows from Theorem 2.3 and [14, Lemma 3.1] that X^i is a local $\mathbb{P}_x$-martingale. Since $\mathscr{E}(X^i) = 1 + \mathscr{E}(X^i)_- \bullet X^i$ by definition of the stochastic exponential, the assertion now follows from [13, I.4.34], because $\mathscr{E}(X^i)_-$ is locally bounded.
$\Rightarrow$: As $\kappa_j = 0$ for $j = m+1, \ldots, d$ and X^j_- is nonnegative for $j = 1, \ldots, m$, [14, Lemma 3.1] and Theorem 2.3 yield that $\int_{\{|\xi_i|>1\}} |\xi_i| \kappa_0(d\xi) < \infty$ and

$$\int_{\{|\xi_i|>1\}} |\xi_i| \kappa_j(d\xi) X^j_- < \infty, \quad 1 \leq j \leq m, \tag{4.3}$$

up to a $d\mathbb{P}_x \otimes dt$-null set on $\Omega \times \mathbb{R}_+$ for any $x \in D$. Let $k \in \{1, \ldots, d\}$ and denote by e_k the k-th unit vector. By Lemma B.1, there exists $t_k > 0$ such that $\frac{1}{2} \geq \mathbb{P}_{e_k}(|X^k_{t-} - 1| > \frac{1}{2}) \geq \mathbb{P}_{e_k}(X^k_{t-} \leq \frac{1}{2})$ and hence $\mathbb{P}_{e_k}(X^k_{t-} > \frac{1}{2}) \geq \frac{1}{2}$ for all $t \leq t_k$. Consequently it follows from (4.3) that $\int_{\{|\xi_i|>1\}} |\xi_i| \kappa_k(d\xi) < \infty$ as claimed. We now turn to (4.2), which is well-defined by (4.1). Set

$$\widetilde{\beta}^i_j := \beta^i_j + \int_{D \setminus \{0\}} (\xi_i - h_i(\xi)) \kappa_j(d\xi), \quad 0 \leq j \leq d.$$

Again by [14, Lemma 3.1] and Theorem 2.3, we have

$$\widetilde{\beta}^i_0 + \sum_{j=1}^d \widetilde{\beta}^i_j X^j_- = 0, \tag{4.4}$$

up to a $d\mathbb{P}_x \otimes dt$-null set on $\Omega \times \mathbb{R}_+$ for all $x \in D$. Suppose $|\widetilde{\beta}^i_0| > 0$. Then by (4.4) we have $\sum_{j=1}^d |\widetilde{\beta}^i_j| > 0$. Lemma B.1 shows that there exists $t_0 > 0$ such that $\mathbb{P}_0(\|X_{t-}\| > |\widetilde{\beta}^i_0|/(2\sum_{j=1}^d |\widetilde{\beta}^i_j|)) \leq \frac{1}{2}$ and therefore

$$\begin{aligned}\mathbb{P}_0(\big|\widetilde{\beta}^i_0 + \sum_{j=1}^d \widetilde{\beta}^i_j X^j_{t-}\big| > 0) &\geq \mathbb{P}_0\left(\big|\widetilde{\beta}^i_0 + \sum_{j=1}^d \widetilde{\beta}^i_j X^j_-\big| > \frac{|\widetilde{\beta}^i_0|}{2}\right)\\ &\geq \mathbb{P}_0\left(\|X_{t-}\| < \frac{|\widetilde{\beta}^i_0|}{2\sum_{j=1}^d |\widetilde{\beta}^i_j|}\right) \geq \frac{1}{2},\end{aligned}$$

for all $t \leq t_0$. Hence

$$\int_0^{t_0} \int 1_{\{|\widetilde{\beta}^i_0 + \sum_{j=1}^d \widetilde{\beta}^i_j X^j_-|>0\}} d\mathbb{P}_0 dt \geq \frac{t_0}{2},$$

which contradicts (4.4). Thus $\widetilde{\beta}^i_0 = 0$. Now suppose $|\widetilde{\beta}^i_k| > 0$ for some $k \in \{1, \dots, d\}$. By Lemma B.1 there is $t_k > 0$ with $\mathbb{P}_{e_k}(\|X_{t-} - e_k\| < |\widetilde{\beta}^i_k|/(4\sum_{j=1}^d |\widetilde{\beta}^i_j|)) \geq \frac{1}{2}$. Therefore

$$\begin{aligned}\mathbb{P}_{e_k}(\big|\widetilde{\beta}^i_0 + \sum_{j=1}^d \widetilde{\beta}^i_j X^j_{t-}\big| > 0) &\geq \mathbb{P}_{e_k}\left(\big|\sum_{j=1}^d \widetilde{\beta}^i_j X^j_-\big| > \frac{|\widetilde{\beta}^i_k|}{2}\right)\\ &\geq \mathbb{P}_{e_k}\left(\|X_{t-} - e_k\| < \frac{|\widetilde{\beta}^i_k|}{4\sum_{j=1}^d |\widetilde{\beta}^i_j|}\right) \geq \frac{1}{2},\end{aligned}$$

for all $t \leq t_k$, which yields a contradiction to (4.4) as above. Hence $\widetilde{\beta}^i_k = 0$ and we are done. □

The nonnegativity of $\mathscr{E}(X^i)$ can also be characterized completely in terms of the parameters of X.

Lemma 4.2. *Let $i \in \{1, \dots, d\}$. Then $\mathscr{E}(X^i)$ is $\mathbb{P}_x$-a.s. nonnegative for all $x \in D$ if and only if*

$$\kappa_j(\{\xi \in D : \xi_i < -1\}) = 0, \quad 0 \leq j \leq m. \tag{4.5}$$

Proof. Fix $x \in D$ and let $T > 0$. By [13, I.4.61], $\mathscr{E}(X^i)$ is $\mathbb{P}_x$-a.s. nonnegative on $[0, T]$ if and only if $\mathbb{P}_x(\exists\, t \in [0, T] : \Delta X^i_t < -1) = 0$. By [13, II.1.8] and Theorem

2.3 this in turn is equivalent to

$$\begin{aligned} 0 &= \mathbb{E}_x\left(\sum_{t\le T} 1_{(-\infty,-1)}(\Delta X^i_t)\right) \\ &= \mathbb{E}_x\left(1_{(-\infty,-1)}(\xi_i) * \mu^X_T\right) \\ &= \mathbb{E}_x\left(1_{(-\infty,-1)}(\xi_i) * \nu_T\right) \\ &= T\kappa_0(\{\xi\in D:\xi_i<-1\}) + \sum_{j=1}^m \kappa_j(\{\xi\in D:\xi_i<-1\})\int_0^T \mathbb{E}_x(X^j_{t-})dt. \end{aligned} \tag{4.6}$$

$\Leftarrow$: Apparently, (4.5) implies that (4.6) hold for every T, which yields the assertion. $\Rightarrow$: Since X^j is nonnegative for $j=1,\dots,m$, (4.6) implies that $\kappa_0(\{\xi\in D:\xi_i<-1\})=0$ and $\kappa_j(\{\xi\in D:\xi_i<-1\})\int_0^T \mathbb{E}_x(X^j_{t-})dt = 0$ for all $x\in D$. For $j\in\{1,\dots,m\}$ it follows from Lemma B.1 that there exists $t_j>0$ such that $\mathbb{P}_{e_j}(|X^j_{t-}|>\frac{1}{2})\ge\frac{1}{2}$ for all $t\le t_j$ and hence $\int_0^T \mathbb{E}_{e_j}(X^j_{t-})dt \ge \frac{t_j\wedge T}{4}$. Consequently, $\kappa_j(\{\xi\in D:\xi_i<-1\})=0$. □

Every positive local martingale of the form $M=\mathscr{E}(X^i)$ is a true martingale for processes X^i with independent increments by [15, Proposition 3.12]. In general, this does not hold true for affine processes as exemplified by [15, Example 3.11], where the following *necessary* condition is violated.

Lemma 4.3. *Let $i\in\{1,\dots,d\}$ such that $M=\mathscr{E}(X^i)$ is $\mathbb{P}_x$-a.s. nonnegative for all $x\in D$. If M is a local $\mathbb{P}_x$-martingale for all $x\in D$, the parameters $(\alpha^\star,\beta^\star,0,\kappa^\star)$ given by*

$$\alpha^\star_j := \alpha_j, \qquad 0\le j\le m, \tag{4.7}$$

$$\beta^\star_j := \beta_j + \alpha^{\cdot i}_j + \int_{D\setminus\{0\}} (\xi_i h(\xi))\kappa_j(d\xi), \quad 0\le j\le d, \tag{4.8}$$

$$\kappa^\star_j(d\xi) := (1+\xi_i)\kappa_j(d\xi), \qquad 0\le j\le d, \tag{4.9}$$

are admissible. If M is a true $\mathbb{P}_x$-martingale for all $x\in D$, the corresponding affine process $(X,\mathbb{P}^\star_x)_{x\in D}$ is conservative.

Proof. The first part of the assertion follows from Lemmas 4.1 and 4.2 as in the proof of [15, Lemma 3.5]. Let M be a true martingale for all $x\in D$. Then for every $x\in D$, e.g. [4] shows that there exists a probability measure $\mathbb{P}^M_x \overset{\text{loc}}{\ll} \mathbb{P}_x$ on $(\mathbb{D}^d,\mathscr{D}^d,(\mathscr{D}^d_t))$ with density process M. Then the Girsanov-Jacod-Memin theorem as in [14, Lemma 5.1] yields that X admits affine $\mathbb{P}^M_x$-characteristics as in (2.5)-(2.7) with $(\alpha,\beta,0,\kappa)$ replaced by $(\alpha^\star,\beta^\star,0,\kappa^\star)$. Since $\mathbb{P}^M_x|_{\mathscr{D}_0}=\mathbb{P}_x|_{\mathscr{D}_0}$ implies $\mathbb{P}^M_x(X_0=x)=1$, we have $\mathbb{P}^M_x=\mathbb{P}^\star_x$ by Theorem 2.3 . In particular, the transition function $p^\star_t(x,d\xi)$ of $(X,\mathbb{P}^\star_x)_{x\in D}$ satisfies $1=\mathbb{P}^M_x(X_t\in D)=\mathbb{P}^\star_x(X_t\in D)=p^\star_t(x,D)$, which completes the proof. □

If $M=\mathscr{E}(X^i)$ is only a local martingale, the affine process $(X,\mathbb{P}^\star_x)_{x\in D}$ does not necessarily have to be conservative (see [15, Example 3.11]). A careful inspection of the proof of [15, Theorem 3.1] reveals that conservativeness of $(X, \mathbb{P}^\star_x)_{x\in D}$ is also a *sufficient* condition for M to be a martingale. Combined with Lemma 4.1 and Theorem 3.3 this in turn allows us to provide the following deterministic necessary and sufficient conditions for the martingale property of M in terms of the parameters of X.

Theorem 4.4. *Let $i \in \{1, \dots, d\}$ such that $\mathscr{E}(X^i)$ is $\mathbb{P}_x$-a.s. nonnegative for all $x \in D$. Then we have equivalence between:*

(i) *$\mathscr{E}(X^i)$ is a true $\mathbb{P}_x$-martingale for all $x \in D$.*
(ii) *$\mathscr{E}(X^i)$ is a local $\mathbb{P}_x$-martingale for all $x \in D$ and the affine process corresponding to the admissible parameters $(\alpha^\star, \beta^\star, 0, \kappa^\star)$ given by* (4.7)-(4.9) *is conservative.*
(iii) (4.1) *and* (4.2) *hold and $g = 0$ is the only $\mathbb{R}^m_-$-valued local solution of*

$$\partial_t g(t) = R^\star_I(g(t), 0), \quad g(0) = 0, \tag{4.10}$$

where $R^\star$ is given by (2.4) *with $(\alpha^\star, \beta^\star, 0, \kappa^\star)$ instead of $(\alpha, \beta, \gamma, \kappa)$.*

Proof. (i) $\Rightarrow$ (ii): This is shown in Lemma 4.3.
(ii) $\Rightarrow$ (iii): This follows from Lemma 4.1 and Theorem 3.3.
(iii) $\Rightarrow$ (i): By (4.1), (4.2) and Lemma 4.2, Assumptions 1-3 of [15, Theorem 3.1] are satisfied. Since we consider time-homogeneous parameters here, Condition 4 of [15, Theorem 3.1] also follows immediately from (4.1). The final Condition 5 of [15, Theorem 3.1] is only needed in [15, Lemma 3.5] to ensure that a semimartingale with affine characteristics relative to $(\alpha^\star, \beta^\star, 0, \kappa^\star)$ exists. In view of the first part of Lemma 4.3, Theorem 3.3 and Theorem 2.3 it can therefore be replaced by requiring that 0 is the unique $\mathbb{R}^m_-$-valued solution to (4.10). The proof of [15, Theorem 3.1] can then be carried through unchanged. □

Remark 4.5. (i) In view of [15, Lemma 2.7], $\widetilde{M} := \exp(X^i)$ can be written as $\widetilde{M} = \exp(X^i_0)\mathscr{E}(\widetilde{X}^i)$ for the $d+1$-th component of the $\mathbb{R}^m_+ \times \mathbb{R}^{n+1}$-valued affine process $(X, \widetilde{X}^i)$ corresponding to the admissible parameters $(\widetilde{\alpha}, \widetilde{\beta}, 0, \widetilde{\kappa})$ given by $(\widetilde{\alpha}_{d+1}, \widetilde{\beta}_{d+1}, \widetilde{\kappa}_{d+1}) = (0, 0, 0)$ and

$$(\widetilde{\alpha}_j, \widetilde{\beta}_j, \widetilde{\kappa}_j(G)) := \left(\begin{pmatrix} \alpha_j & \alpha^{\cdot i}_j \\ \alpha^{i\cdot}_j & \alpha^{ii}_j \end{pmatrix}, \begin{pmatrix} \beta_j \\ \widetilde{\beta}^{d+1}_j \end{pmatrix}, \int_{D\setminus\{0\}} 1_G(\xi, e^{\xi_i} - 1)\kappa_j(d\xi) \right)$$

for $G \in \mathscr{B}^{d+1}$, $j = 0, \dots, d$, and

$$\widetilde{\beta}^{d+1}_j = \beta^i_j + \frac{1}{2}\alpha^{ii}_j + \int_{D\setminus\{0\}} (h_i(e^{\xi_i} - 1) - h_i(\xi))\kappa_j(d\xi).$$

This allows to apply Theorem 4.4 in this situation as well.
(ii) Conservativeness of $(X, \mathbb{P}^\star_x)_{x\in D}$ and uniqueness for solutions of ODE (3.3) can be ensured by requiring the moment condition (3.5) for $\kappa^\star_j$. The implication (iii) $\Rightarrow$ (i) in Theorem 4.4 therefore leads to the easy-to-check sufficient criterion [15, Corollary 3.9] for the martingale property of M.
(iii) By Remark 3.4 (iv) we know that in the case $m = 1$, $(X, \mathbb{P}^\star_x)_{x\in D}$ is conservative if and only if either (3.5) holds for $\kappa^\star_j$ or equation (3.6) holds for $R^\star_1$.

Together with Remark (i), this leads to the necessary and sufficient condition for the martingale property of ordinary exponentials $\exp(X^i)$ obtained in [18, Theorem 2.5].

Appendix A. ODE comparison results in non-Lipschitz setting

Let C be a closed convex proper cone with nonempty interior C° in a normed vector space $(E, \|\ \|)$. The partial order induced by C is denoted by $\leqq$. For $x, y \in E$, we write $x \ll y$ if $y - x \in C^\circ$. We denote by C^* the dual cone of C. Let D_g be a set in E. A function $g\colon D_g \to E$ is called *quasimonotone increasing*, in short *qmi*, if for all $l \in C^*$, and $x, y \in D_g$

$$(x \leqq y,\ l(x) = l(y)) \Rightarrow (l(g(x)) \leq l(g(y))).$$

The next lemma is a special case of Volkmann's result [23, Satz 1].

Lemma A.1. *Let $0 < T \leq \infty$, $D_f \subset E$, and $f\colon [0,T) \times D_f \to E$ be such that $f(t,\cdot)$ is qmi on D for all $t \in [0,T)$. Let $\zeta, \eta : [0,T) \to D_f$ be curves that are continuous on $[0,T)$ and differentiable on $(0,T)$. Suppose $\zeta(0) \gg \eta(0)$ and $\dot\zeta(t) - f(t,\zeta(t)) \gg \dot\eta(t) - f(t,\eta(t))$ for all $t \in (0,T)$. Then $\zeta(t) \gg \eta(t)$ for all $t \in [0,T)$.*

A function $g : [0,T) \times D_g \to E$ is called *locally Lipschitz*, if for all $0 < t < T$ and for all compact sets $K \subset D_g$ we have

$$L_{t,K}(g) := \sup_{0<\tau<t,\ x,y\in K: x\neq y} \frac{\|g(\tau,x) - g(\tau,y)\|}{\|x-y\|} < \infty$$

where $L_{t,K}(g)$ is usually called the Lipschitz constant.

We now use Lemma A.1 to prove the following general comparison result.

Proposition A.2. *Let T, D_f, and f be as in Lemma A.1. Suppose, moreover, that D_f has a nonempty interior and f is locally Lipschitz on $[0,T) \times D_f^\circ$. Let $\zeta, \eta : [0,T) \to D_f$ be curves that are continuous on $[0,T)$, differentiable on $(0,T)$, and satisfy the conditions*

(i) $\eta(t) \in D_f^\circ$

(ii) $\dot\zeta(t) - f(t,\zeta(t)) \geqq \dot\eta(t) - f(t,\eta(t))$

(iii) $\zeta(0) \geqq \eta(0)$

for all $t \in [0,T)$. Then $\zeta(t) \geqq \eta(t)$ for all $t \in [0,T)$.

Proof. Fix $t_0 \in [0,T)$. Since η is continuous, the image S of the segment $[0,t_0]$ under the map η is a compact subset of D_f°. Let $\delta > 0$ be such that the closed δ-neighborhood S_δ of S is contained in D_f°. By the local Lipschitz continuity of f on D_f°, there exists a constant $L > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \tag{A.1}$$

for any $t \in [0,t_0]$ and $x, y \in S_\delta$. Let $c \in C^\circ$ be such that $\|c\| = 1$ and let d_c denote the distance from c to the boundary ∂C of C. For $\varepsilon > 0$, we set $h_\varepsilon(t) := \varepsilon e^{2Lt/d_c} c$. If $\varepsilon \leq e^{-2Lt_0/d_c}\delta$, then $\eta(t) - h_\varepsilon(t) \in S_\delta$ for any $t \in [0,t_0]$, and (A.1) gives

$$\|f(t,\eta(t) - h_\varepsilon(t)) - f(t,\eta(t))\| \leq L\|h_\varepsilon(t)\|, \quad t \in [0,t_0]. \tag{A.2}$$

Since C is a cone, the distance from $Lh_\varepsilon(t)/d_c$ to ∂C is equal to $L\varepsilon e^{2Lt/d_c} = L\|h_\varepsilon(t)\|$. In view of (A.2), it follows that

$$Lh_\varepsilon(t)/d_c \geqq f(t, \eta(t) - h_\varepsilon(t)) - f(t, \eta(t))$$

and hence

$$-\dot{h}_\varepsilon(t) = -2Lh_\varepsilon(t)/d_c \ll f(t, \eta(t) - h_\varepsilon(t)) - f(t, \eta(t)), \quad t \in [0, t_0], \tag{A.3}$$

for ε small enough. This implies that

$$\dot{\zeta}(t) - f(t, \zeta(t)) \geqq \dot{\eta}(t) - f(t, \eta(t)) \gg \dot{\eta}(t) - \dot{h}_\varepsilon(t) - f(t, \eta(t) + h_\varepsilon(t)).$$

Applying Lemma A.1 to the functions $\zeta(t)$ and $\eta(t)+h_\varepsilon(t)$ yields $\zeta(t) \gg \eta(t)+h_\varepsilon(t)$, for all $t \in [0, t_0]$. Now letting $\varepsilon \to 0$ yields the required inequality for all $t \in [0, t_0]$. This proves the assertion, because $t_0 < T$ can be chosen arbitrarily. □

If we consider the differential equation

$$\dot{\xi} = f(t, \xi(t)), \quad \xi(0) = u \in D_f, \tag{A.4}$$

Proposition A.2 allows the following immediate conclusion, which is the key tool for proving Proposition 3.2 and in turn Theorem 3.3.

Corollary A.3. *Let T, D_f and f be as in Lemma A.2. Suppose further that equation* (A.4) *gives rise to a global solution $\psi^\circ(t,u)\colon \mathbb{R}_+ \times D_f^\circ \to D_f^\circ$. Let $u_2 \in D_f^\circ$ and let $\xi\colon [0,T) \to D_f$ be a solution of* (A.4) *such that $\xi(0) = u_1 \geqq u_2$. Then $\xi(t) \geqq \psi^\circ(t, u_2)$, for all $t \in [0, T)$.*

Appendix B. A lemma on stochastic continuity

Lemma B.1. *Let $(X, \mathbb{P}_x)_{x \in D}$ be a conservative, stochastically continuous affine process and fix $x \in D$. Then for any $\varepsilon, \delta > 0$ there exists t_x such that*

$$\mathbb{P}_x(|X_{t-} - x| > \varepsilon) < \delta, \quad \forall t \le t_x. \tag{B.1}$$

Proof. Fix $x \in D$. Since $p_t(x, \cdot) \to p_0(x, \cdot)$ weakly, it follows from [1, Satz 30.12] that (B.1) holds for X_t instead of X_{t-}. Now notice that the absolute continuity of the characteristics of X (cf. Theorem 2.3) combined with [13, II.2.9 and I.2.25] yields $\Delta X_t = 0$, $\mathbb{P}_x$-a.s. Hence

$$\begin{aligned} \mathbb{P}_x(|X_{t-} - x| > \varepsilon) &\le \mathbb{P}_x(|X_t - x| > \varepsilon) + \mathbb{P}_x(|\Delta X_t| > 0) \\ &= \mathbb{P}_x(|X_t - x| > \varepsilon) \\ &< \delta, \end{aligned}$$

for all $t \le t_x$. This proves the assertion. □

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