

A note on the affine transform formula for affine diffusions

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Abstract

In this note a proof for the exponentially affine form of the conditional moment generating function of affine diffusion processes is provided, which confirms a recent result obtained jointly with Damir Filipović in the review article "Affine Diffusion processes, theory and applications" (2009). The present proof avoids the use of the Markov property completely, and employs elementary Martingale arguments in conjunction with properties of the involved Riccati equations instead.

1 Introduction

An affine process on some state-space $D \subseteq \mathbb{R}^d$ is affine, if it is a stochastically continuous Markov process $X^x(t)$ whose characteristic function is of exponentially affine form in the state variable. That is, there exist functions $\phi(t, u)$ and $\psi(t, u)$ such that

$$\mathbb{E}[e^{u^\top X^x(t)}] = e^{\phi(t, u) + \psi(t, u)^\top x}, \quad (1.1)$$

holds for all $u \in i\mathbb{R}^d$, $x \in D$ and $t \geq 0$.

In the case of canonical state-spaces $D = \mathbb{R}_+^m \times \mathbb{R}^n$ this class of Markov processes is well understood, see [5], and for positive semi-definite matrices S_d^+ , a full characterization is available as well – with the subtle difference that the definition of the affine property (1.1) is in terms of the Laplace-transform, see [3].

The class of affine processes provides multivariate continuous time stochastic models which are analytically tractable, because their Fourier-transform is of a particularly simple form, and can, in addition, be derived by solving a system of Riccati ODEs, (see eqs. (1.4)–(1.5) below). Sometimes, the affine transform formula is available in closed form, for instance in Heston's celebrated stochastic volatility model [8], or in matrix valued Wishart models [2, 3], which are natural multivariate generalizations of square Bessel processes.

An interesting question, concerning property (1.1), is its validity beyond its natural domain, namely away from the imaginary axis $i\mathbb{R}^d$. The importance of this question maybe best displayed by the method of pricing exponentially bounded contingent claims $f(X_T)$ in mathematical finance via Fourier Inversion ([6, 9] and the references to original sources given therein). The latter requires¹ that for certain $v \in \mathbb{R}^d$,

$$\mathbb{E}[e^{v^\top X^x(t)}] < \infty, \quad \text{for all } x \in D$$

and that, in addition, the affine transform formula (1.1) holds as well for the complex line $L_v := \{u = v + i\lambda \mid \lambda \in \mathbb{R}^d\}$. In fact, if f can be expressed as

$$f(x) = \int_{\mathbb{R}^d} e^{(u+i\lambda)^\top x} \tilde{f}(y) dy$$

with some integrable function \tilde{f} , then an application of Fubini's theorem yields

$$\mathbb{E}[f(X_T^x) \mid \mathcal{F}_t] = \int_{\mathbb{R}^d} e^{\phi(T-t, v+i\lambda) + \psi(T-t, v+i\lambda)^\top X^x(t)} \tilde{f}(\lambda) d\lambda \quad (1.2)$$

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¹ v corresponds to the damping factor, which makes $x \mapsto e^{-v^\top x} f(x)$ integrable.

whenever $(\phi(t, u + i\lambda), \psi(t, u + i\lambda))$ are well defined on L_v . Formula 1.2 can be interpreted as the \mathcal{F}_t -conditional arbitrage-free price of a contingent claim $f(X_T)$, under the assumption that $X^x(t)$ is modeled under the risk-neutral measure, i.e., it is a martingale (or components of it, depending which components model traded assets), and if the interest rate is assumed to be zero.²

In this note a new proof for the exponential affine form of the conditional moment generating function of an affine diffusion process $X^x(t)$ on canonical state-spaces $\mathbb{R}_+^m \times \mathbb{R}^n$ (denoted by D from now on) is given. To this end we recall [6, Lemma 6.2]: Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ carrying a d -dimensional Brownian motion $W(t)$, the process $X^x(t)$ can be realized as strong solution³ of the SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x \in D,$$

where both drift $b(x)$ and diffusion coefficient $a(x) = \sigma(x)\sigma^\top(x)$ are affine functions in the state variable x . That is,

$$\begin{aligned} a(x) &= a + \sum_{i=1}^d x_i \alpha_i \\ b(x) &= b + \sum_{i=1}^m x_i \beta_i = b + \mathcal{B}x \end{aligned} \tag{1.3}$$

for some symmetric positive semi-definite $d \times d$ -matrices a and α_i , and d -vectors b and β_i , where we denote by

$$\mathcal{B} = (\beta_1, \dots, \beta_d)$$

the $d \times d$ -matrix with i -th column vector β_i , $1 \leq i \leq d$.⁴

In addition, one knows that for any $u \in i\mathbb{R}^d$, the characteristic exponents $\phi(t, u)$ and $\psi(t, u)$ are the unique global solutions of the $(d+1)$ -dimensional Riccati equations

$$\partial_t \phi(t, u) = F(\psi(t, u)), \quad \phi(0, u) = 0 \tag{1.4}$$

$$\partial_t \psi(t, u) = R(\psi(t, u)), \quad \psi(0, u) = u. \tag{1.5}$$

where F , and R are quadratic functions with coefficients given in terms of drift and diffusion parameters as follows

$$F(u) = u^\top a u + b^\top u, \quad R_i(u) = u^\top \alpha_i u + \beta_i^\top u, \quad i = 1, \dots, d. \tag{1.6}$$

The conditional characteristic function of $X^x(t)$ is then given as

$$\mathbb{E} \left[e^{u^\top X^x(T)} \mid \mathcal{F}_t \right] = e^{\phi(T-t, u) + \psi(T-t, u)^\top X^x(t)} \tag{1.7}$$

for all $u \in i\mathbb{R}^d$, $t \leq T$ and $x \in D$.

In fact, by the very definition of ϕ and ψ , and by an application of Itô's formula, to

$$M(t; u, x) := e^{\phi(T-t, u) + \psi(T-t, u)^\top X^x(t)} \tag{1.8}$$

one obtains

$$\frac{dM(t; u, x)}{M(t; u, x)} = I(t) dt + \psi(T-t, u)^\top \sigma(X^x(t)) dW(t), \quad t \leq T,$$

with

$$\begin{aligned} I(t) &= -\partial_T \phi(T-t, u) - \partial_T \psi(T-t, u)^\top X^x(t) \\ &\quad + \psi(T-t, u)^\top b(X^x(t)) + \frac{1}{2} \psi(T-t, u)^\top a(X^x(t)) \psi(T-t, u) = 0 \end{aligned} \tag{1.9}$$

²This assumption is just for simplicity of demonstration.

³Modulo an affine transformation of the state-space, this result is an application of [14, Theorem 1]. The existence of strong solutions is, however, of no importance to this paper.

⁴Further parametric restrictions are not essential for this note, even though they are implicitly used in Lemma 2.3. They can be found in [6, Theorem 3.2], In the more general context of jump-diffusions they are provided by [5, Definition 2.6] (in slightly different notation).

which vanishes identically because of (1.4)–(1.5). Hence the right side of (1.7) is a local martingale on $[0, T]$, and since $u \in i\mathbb{R}^d$, the affine property (1.1) implies that the real parts $\Re\phi(t, u) \leq 0$ $\Re\psi_i(t, u) \leq 0$, for $i = 1, \dots, m$.⁵ Therefore $M(t; u, x)$ is a uniformly bounded local martingale. Hence it is a true martingale with finite value $M(T; u, x) = e^{u^\top X^x(T)}$, whence formula (1.7) immediately follows.

Remark 1.1. In investigating the validity of the conditional affine transform formula (1.7) for $u \notin i\mathbb{R}^d$, along the above lines, the following issues arise:

- (i) First, to be able to define the left side of (1.7), we need to answer the following question:
Is $\mathbb{E}[|e^{u^\top X(T)}|] < \infty$?
- (ii) Secondly, concerning the right side of (1.7), we are led to investigate:
 - (a) Are the exponents $\phi(t, u)$, $\psi(t, u)$ well defined on the interval $t \in [0, T]$?
 - (b) Is $M(t; u, x)$ a true martingale on $[0, T]$?

The full validity of the affine transform formula for the unconditional moment generating function of $X^x(t)$, that is (1.1) for $u \in \mathbb{R}^d$, was revealed by [7] in the mean reverting regime under some specific assumptions on the drift and diffusion coefficients, namely the class of term structure models proposed by Dai and Singleton [4] was covered. In [6], the results were extended to the whole class of affine diffusions, in particular it was shown that the mean reverting assumption was unnecessary, thereby including affine models such as Heston's. The main statement of that work, [6, Theorem 3.3], is recalled in Theorem 2.1 below in a slightly modified way⁶. Concerning the validity of (1.1) it was obtained, that the left side is well defined if and only the right side is (Theorem 2.1 (ii)): The u -th exponential moment of $X^x(t)$ is finite for all $x \in D$ if and only if the Riccati equations for $\phi(s, u)$ and $\psi(s, u)$ are solvable until t .⁷ Hence, the problems of Remark 1.1(i) and (ii)a are fully settled; The paper's concern is the remaining issue (ii)b.

The review [6] also sketches a proof for the validity of the affine transform formula for the conditional moment generating function using the Markov property (this is Theorem 3.3 (4), see also the modified statement 2.1 (iv) and Remark 2.2). This note reconfirms the validity of the assertion of Theorem 2.1 (iv) by using the martingale argument as above instead. If this proof is more accessible from a didactic point of view, then this note serves its purpose.

It should be noted here that sufficient conditions for the validity of the conditional affine transform formula are also given in [10, Theorem 5.1], they however apply to real moments (that is, $\mathbb{E}[e^{p^\top X^x(T)} | \mathcal{F}_t]$, $p \in \mathbb{R}^d$) only. For related literature on moment explosion in affine models we refer to [1, 11] and the references given therein.

2 Statement of the main theorem

We start with some preliminary notation and recall some elementary facts, which can be found in greater detail in [6, section 2]. In view of the polynomial form of F and R as defined in (1.6), the generalized Riccati differential equations (1.4)–(1.5) can be solved for arbitrary complex initial data $u \in \mathbb{C}$. We solve this equations forward in time, i.e., for $t \geq 0$. The standard existence and uniqueness result for ODEs asserts that there exists a maximal life time $t_+(u) > 0$ for the solutions $\phi(t, u)$, $\psi(t, u)$ of (1.4)–(1.5) such that for $t \rightarrow t_+(u)$, the solutions explodes (its norm tends to infinity). The associated maximal domain $\mathcal{D}_{\mathbb{C}} := \{(t, u) \in \mathbb{R}_+ \times \mathbb{C} \mid t < t_+(u)\}$ is open in $\mathbb{R}_+ \times \mathbb{C}$. For $\tau > 0$, the associated τ -section is denoted by

$$\mathcal{D}_{\mathbb{C}}(\tau) := \{u \in \mathbb{C} \mid (\tau, u) \in \mathcal{D}_{\mathbb{C}}\}.$$

If we solve for real data u only, the associated domains are defined similarly and denoted by $\mathcal{D}_{\mathbb{R}}$, $\mathcal{D}_{\mathbb{R}}(\tau)$.

Let $p \in \mathbb{R}^d$. If the p -th exponential moment

$$\mathbb{E}[e^{p^\top X^x(t)}] < \infty$$

⁵ $\Re\psi_i(t, u) = 0$ for $i \geq m + 1$ and $u \in i\mathbb{R}^d$, because the equation for ψ_i is linear in that case.

⁶See Remark 2.2

⁷Hence beyond t , by the standard existence result for ODEs

then clearly the same holds for the whole complex line $p + i\lambda$, $\lambda \in \mathbb{R}^d$. We introduce the complex tubification of a real domain $U \subset \mathbb{R}^d$ as follows

$$\mathcal{S}(U) := \{u + i\lambda \mid u \in U, \lambda \in \mathbb{R}^d\}.$$

Theorem 2.1. *Suppose X is a conservative affine diffusion process on $D := \mathbb{R}_+^m \times \mathbb{R}^n$. Let $\tau > 0$. Then*

- (i) $\mathcal{S}(\mathcal{D}_{\mathbb{R}}(\tau)) \subset \mathcal{D}_{\mathbb{C}}(\tau)$
- (ii) $\mathcal{D}_{\mathbb{R}}(\tau) = M_{\mathbb{R}}(\tau)$ where

$$M_{\mathbb{R}}(\tau) = \left\{ u \in \mathbb{R}^d \mid \mathbb{E} \left[e^{u^\top X^x(\tau)} \right] < \infty \text{ for all } x \in D \right\}$$

and it holds that for all $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(\tau))$, $x \in D$ and $t \leq \tau$ we have that

$$\mathbb{E}[e^{u^\top X^x(t)}] = e^{\phi(t,u) + \psi(t,u)^\top x}. \quad (2.1)$$

- (iii) $\mathcal{D}_{\mathbb{R}}(\tau)$ and $\mathcal{D}_{\mathbb{R}}$ are convex sets.

Moreover, for all $0 \leq t \leq T$ and $x \in D$,

- (iv) (1.7) holds for all $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(T))$
- (v) (1.7) holds for all $u \in \mathbb{C}^m \times i\mathbb{R}^n$
- (vi) $M_{\mathbb{R}}(t) \supseteq M_{\mathbb{R}}(T)$.

Remark 2.2. • This note's aim is to provide a new and self-contained proof of statement (iv), whereas the rest of the statements of Theorem 2.1 are explained in full detail in [6, Proof of Theorem 3.3].

- There are a few more or less important changes in the statement, compared to [6, Theorem 3.3].
 - Point (ii) is extended by the assertion concerning the validity of unconditional affine transform formula (2.1). In fact, the proof of [6, Theorem 3.3 (2)] contains this assertion, anyway. We base our alternative proof of (iv) on both parts of (ii), and therefore the present reformulation makes this note logically consistent.
 - In point (iv) we remove a typo from the original statement [6, Theorem 3.3 (4)], in that we impose the condition $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(T))$ instead of $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(T-t))$. In view of Theorem 2.1 (ii) this is the best we can do, if we want the left side of (1.7) to be finite almost surely, see also Remark 1.1 (i). Note however that for fixed t , the right side of (1.7) is clearly finite, almost surely, whenever $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(T-t))$. We note that the discovered typo has no impact on the rest of the paper [6], particularly the bond and option formulae presented on pp. 136–139.
 - A notational change concerning the domain $M_{\mathbb{R}}(t)$ is made, in order not to confuse it with the martingale $M(t; u, x)$ defined in (1.8). The original notation is $M(t)$.

We start with recalling the following:

Lemma 2.3. *Let $u, \lambda \in \mathbb{R}^d$. Then $t_+(u + i\lambda) \geq t_+(u)$ and for all $t < t_+(u)$ we have that $\Re(\phi(t, u + i\lambda)) \leq \phi(t, u)$ and $\Re(\psi_i(t, u + i\lambda)) \leq \psi_i(t, u)$ for all $i = 1, \dots, d$.*

Proof. The first assertion $t_+(u + i\lambda) \geq t_+(u)$ is just a reformulation of Theorem 2.1(i). Since α_i are positive definite for each $i = 1, \dots, m$, we have that for $u, \lambda \in \mathbb{R}^d$

$$\Re(R_i(u + i\lambda)) = u^\top \alpha_i u - \lambda^\top \alpha_i \lambda + \beta_i^\top u \leq u^\top \alpha_i u + \beta_i^\top u = R_i(u)$$

and similarly, $\Re(F(u + i\lambda)) \leq F(u)$. Furthermore for $j = m+1, \dots, d$, we have $\alpha_j = 0$, hence by linearity of R_j we have $\Re(R_j(u + i\lambda)) = R_j(u)$. We conclude that for $i = 1, \dots, d$, we have that

$$\begin{aligned} \partial_t \Re(\psi_i(t, u + i\lambda)) - R_i(\Re(\psi(t, u + i\lambda))) &\leq \partial_t \psi_i(t, u) - R_i(\psi(t, u)) \\ \partial_t \Re(\phi(t, u + i\lambda)) - F(\Re(\phi(t, u + i\lambda))) &\leq \partial_t \phi(t, u) - R(\phi(t, u)) \\ \Re(\phi(0, u + i\lambda)) &= \phi(0, u) = 0 \\ \Re(\psi(0, u + i\lambda)) &= \psi(0, u) = u \end{aligned}$$

hence by a multivariate ODE comparison result such as [13], and by the quasi-monotonicity of the first m components of $R(u_1, \dots, u_m, \psi_{m+1}(t), \dots, \psi_d(t))$, for each t , with respect to the natural cone \mathbb{R}_+^m [12] we obtain the claim. \square

Proof of Theorem 2.1(iv). Let $u \in \mathbb{R}^d$ and pick a finite time horizon T such that $0 < T < t_+(u)$. We claim that for each $\lambda \in \mathbb{R}^d$, and all $x \in D$ the process

$$M(t; x, u + i\lambda) := e^{\phi(T-t, u+i\lambda) + \psi(T-t, u+i\lambda)^\top X^x(t)}$$

is an \mathcal{F}_t -martingale on $[0, T]$.

We start with the case $\lambda = 0$. Then $M(t; x, u)$ is a local martingale by eq. (1.9). Since $M(t; x, u)$ is strictly positive, it is a supermartingale as well. Since any supermartingale with constant expectation is a martingale, it suffices to show that $\mathbb{E}[M_T(x, u)] = M(0; x, u)$. Indeed, we have

$$\mathbb{E}[M(T; x, u)] = \mathbb{E}[e^{\phi(0, u) + \psi(0, u)^\top X(T)^x}] = \mathbb{E}[e^{u^\top X^x(T)}] = e^{\phi(T, u) + \psi(T, u)^\top x} = M(0; x, u)$$

where the last but one equality follows from 2.1. Hence $M(t; x, u)$ is a true martingale on $[0, T]$.

Pick now $\lambda \in \mathbb{R}^d$. By Lemma 2.3 we have that $t_+(u+i\lambda) \geq t_+(u)$, in particular $M(t; x, u+i\lambda)$ is a well defined local martingale on $[0, T]$. Furthermore, for all $t \in [0, T]$ it holds that

$$|M(t; u + i\lambda, x)| \leq e^{\Re\phi(t, u+i\lambda) + \Re(\psi(t, u+i\lambda)^\top X^x(t))} \leq M(t; u, x),$$

where we have applied Lemma 2.3 again. Hence the (in general complex valued) local martingale $M(t; u+i\lambda, x)$ is bounded by a true martingale, namely $M(t; u, x)$. By dominated convergence, see the more detailed statement Lemma 2.4 below, we conclude that $M(t; u+i\lambda, x)$ is a martingale as well. The claim of Theorem 2.1 (iv) now follows from the fact that $M(t; u+i\lambda, x)$ has finite value $M(T; u+i\lambda, x) = e^{u^\top X^x(T)}$, which implies that (1.7) holds. \square

Lemma 2.4. *Let $Z(t)$ be a (possibly complex valued) local martingale and suppose that $|Z(t)| \leq N(t)$, where $N(t)$ is a true martingale. Then $Z(t)$ is a martingale as well.*

Proof. Let $T_1 < T_2 < \dots$ be a localizing sequence for $Z(t)$. Let $T > 0$. By the martingale property of $Z(t \wedge T_n)$, we have that for each $n \geq 1$,

$$\mathbb{E}[N(T \wedge T_n) | \mathcal{F}_t] = N(t \wedge T_n) \tag{2.2}$$

Now, the right side of (2.2) converges to $N(t)$, almost surely.

Concerning the left side of (2.2), we observe that because N is bounded by M , and by the martingale property of M , the family of random variables $\{M(t \wedge T_n)\}_{n \geq 1}$ is uniformly integrable, and therefore $\{M(t \wedge T_n)\}_{n \geq 1}$ is uniformly integrable as well. By Vitali's convergence theorem, we conclude that $M(t \wedge T_n)$ converges to $M(t)$ in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, hence for a subsequence n_k , $\mathbb{E}[N(t \wedge T_{n_k}) | \mathcal{F}_t]$ converges almost surely to $\mathbb{E}[N(t) | \mathcal{F}_t]$, which proves that

$$\mathbb{E}[N(T) | \mathcal{F}_t] = N(t).$$

\square

References

- [1] L. B. G. ANDERSEN AND V. V. PITERBARG, *Moment explosions in stochastic volatility models*, Finance and Stochastics, 11 (2007), pp. 29–50.
- [2] M.-F. BRU, *Wishart Processes*, Journal of Theoretical Probability, 4 (1991), pp. 725–751.
- [3] C. CUCHIERO, AND D. FILIPOVIĆ, AND E. MAYERHOFER, AND J. TEICHMANN, *Affine processes on positive semidefinite matrices*, submitted (2009).
- [4] Q. DAI AND K. J. SINGLETON, *Specification analysis of affine term structure models*, J. of Finance, 55 (2000), pp. 1943–1978.
- [5] D. DUFFIE, D. FILIPOVIĆ, AND W. SCHACHERMAYER, *Affine processes and applications in finance*, Ann. Appl. Probab., 13 (2003), pp. 984–1053.

- [6] D. FILIPOVIĆ, AND E. MAYERHOFER, *Affine Diffusion Processes-Theory and Applications*, Radon Series Comp. Appl. Math 8 (2009), pp. 1–40.
- [7] P. GLASSERMAN AND K.-K. KIM, *Moment explosions and stationary distributions in affine diffusion models*. To appear in *Mathematical Finance*, 2008/2009.
- [8] S. HESTON, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, *Rev. of Financial Studies*.
- [9] F. HUBALEK, J. KALLSEN, AND L. KRAWCZYK, *Variance-optimal hedging for processes with stationary independent increments*, *Ann. Appl. Probab.*, 16 (2006), pp. 853–885.
- [10] J. MUHLE-KARBE, AND J. KALLSEN, *Exponentially Affine Martingales, Affine Measure Changes and Exponential Moments of Affine Processes*, *Stochastic Processes and their Applications* Vol. 120 (2010), No. 2, pp. 163–181. .
- [11] M. KELLER-RESSEL, *Moment explosions and long-term behavior of affine stochastic volatility models*. to appear in *Mathematical Finance*.
- [12] ———, *Affine processes- theory and applications in finance*, PhD thesis Vienna University of Technology, (January, 2009).
- [13] P. VOLKMANN, *Über die Invarianz konvexer Mengen und Differentialgleichungen in einem normierten Raume*, *Math. Ann.* 203, (1973), pp. 201–210.
- [14] T. YAMADA AND S. WATANABE, *On the uniqueness of solutions of stochastic differential equations*, *J. Math. Kyoto Univ.*, 11 (1971), pp. 155–167.