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Complete L^0-normed modules and automatic continuity of monotone convex functions

Michael Kupper and Nicolas Vogelpoth

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Complete L^0–normed modules and automatic continuity of monotone convex functions[*]

Michael Kupper and Nicolas Vogelpoth[†]
Vienna Institute of Finance,[‡]
University of Vienna and
Vienna University of Economics and Business Administration
Heiligenstaedter Strasse 46-48
A-1190 Vienna, Austria

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Abstract

Led by applications to conditional risk measures, we establish continuity results for monotone convex functions from L^0–modules into L^0. The results are generalizations of classical results on continuity of monotone convex functions. We introduce topological L^0–modules of L^p and Orlicz type. We investigate completeness and in the L^p type case we compute the topological dual L^0–module. Applications in terms of risk measures are given.

Key words: Complete L^0–Modules, Locally L^0–Convex Modules, L^0–Convex Functions, Monotone Functions, Continuity, Subdifferentiability, L^p Spaces, Olricz Spaces

1 Introduction

Throughout the recent literature on mathematical finance, there has been a significant effort to establish Fenchel–Moreau type dual representation and subdifferentiability results for conditional risk measures, c.f. [3, 11, 6, 14, 19, 12] and the references therein. When it comes to conditional risk measures, it is often a delicate matter to apply classical convex analysis as the relevant results, such as Hahn–Banach extension and hyperplane separation theorems, are stated for real valued rather than for vector valued functions. Therefore, a module based approach, as proposed in [12], seems advisable. Here, a conditional risk measure is no longer studied as a vector valued function on a vector space but rather as a function from an L^0–module into $L^0 = L^0(\Omega, \mathcal{F}, P)$, the ordered ring of (equivalence classes of) random variables.

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Such a module based approach allows for convex analysis since the relevant results of classical, vector space based convex analysis can be generalized to locally L^0–convex modules, a module analogue of locally convex spaces, c.f. [12].

To establish dual representation results as well as subdifferentiability results for conditional risk measures continuity properties play an important role. A conditional risk measure, however, is defined upon a set of economically rational axioms. Mathematically yielding properties, such as continuity, a priori are not addressed by this set of axioms but remain crucial for convex analysis. Nevertheless, widely accepted axioms of conditional risk measures are that of convexity and monotonicity. In the static case, these two properties implicitly guarantee continuity, c.f. results on continuity of monotone convex functions as in [24, 4]. Moreover, it is even possible to construct subgradients, c.f. [26]. Automatic continuity results in the theory of static risk measures have first been applied in [5] and further developed in [26, 7, 2].

Static risk measures have most extensively been studied as real valued functions on Banach spaces, such as L^p and Orlicz spaces, c.f. [10, 15, 16, 17, 7, 8, 2, 9, 13, 18] and the references therein. Therefore, one aim of the present paper is to introduce module analogues of such spaces. The second aim is to establish automatic continuity results for monotone convex functions on the respective modules. While motivated by financial applications our continuity results are of theoretical nature. In line with this, we illustrate these results by means of conditional risk measures. This is accomplished in the following way.

In the preliminary section 2 we introduce a topology on L^0. We follow the ideas of [12]. Since L^0 is an ordered ring we can define a neighborhood of $0 \in L^0$ as the set of all elements which are bounded by a positive element. It is noteworthy that the collection of such neighborhoods induces a ring topology on L^0 and no longer a vector space topology. The topological ring L^0 enables us to define module topologies on general L^0–modules by means of L^0 valued "norms". Such L^0–modules are called L^0–normed modules and form an important example of locally L^0–convex modules as introduced in [12].

In section 3 we present the main results. These can be summarized as follows. If an L^0–normed module E is complete and also a module lattice, then any proper monotone L^0–convex function $f : E \to \bar{L}^0$ is continuous and subdifferentiable throughout the interior of its effective domain. Here, $\bar{L}^0$ denotes the set of all random variables which may take values in $\mathbb{R} \cup \{\pm\infty\}$. The main results are suited to conditional risk measures and can be viewed as module variants of the results in section 5 of [24] and of proposition 3.1 in [26]. A crucial aspect in the proof of our results is that proper L^0–convex functions $f : E \to \bar{L}^0$ have a certain local property. This property allows to characterize continuity in terms of a specific class of nets rather than in terms of abstract general nets. This class of nets admits a relation to sequences and then to draw from classical results on continuity of monotone convex functions, c.f. [1, 4]. This is outlined in section 5, where we establish the main results.

In section 4, we present important examples of L^0–normed modules. These include modules of L^p and Orlicz type. We show that these modules are complete and in the L^p type case we find the L^0–module of continuous L^0–linear functions. On L^p type modules, we illustrate our results by means of a generalized expected shortfall risk measure. In the case of Orlicz type modules, we have a look at the entropic risk measure.

2 Preliminaries

Let L^0 be the ring of real valued $\mathcal{F}$–measurable random variables on a probability space $(\Omega, \mathcal{F}, P)$. Random variables and measurable sets which coincide almost surely are identified. Equalities and inequalities between random variables are understood in the almost sure sense. We recall that L^0 equipped with the order of almost sure dominance is a lattice ordered ring. Further, $L^0_+ := \{Y \in L^0 \mid Y \geq 0\}$, $L^0_{++} := \{Y \in L^0 \mid Y > 0\}$. $\bar{L}^0$ denotes the set of $\mathcal{F}$–measurable random variables which take values in $\mathbb{R} \cup \{\pm\infty\}$ and $\bar{L}^0_+ := \{Y \in \bar{L}^0 \mid Y \geq 0\}$.

Definition 2.1 *Let E be an L^0–module. A map $\| \cdot \| : E \to L^0_+$ is an L^0–norm (on E) if*

(i) $\|X\| = 0$ *implies* $X = 0$,

(ii) $\|YX\| = |Y|\|X\|$ *for all* $Y \in L^0$ *and* $X \in E$,

(iii) $\|X + X'\| \leq \|X\| + \|X'\|$ *for all* $X, X' \in E$.

An L^0–normed module $(E, \| \cdot \|)$ is an L^0–module E with an L^0–norm $\| \cdot \|$.

Up to normalization, the absolute value $| \cdot | : L^0 \to L^0_+$ is the only L^0–norm on L^0 seen as module over itself. The order of almost sure dominance on L^0 allows to define a natural topology on L^0–normed modules, in particular on L^0 itself. This is outlined below.

Let $(E, \| \cdot \|)$ be an L^0–normed module. Denote $\mathbb{N}(\mathcal{F}) := \{N \in L^0 \mid N(\Omega) \subset \mathbb{N}\}$ and define the ball of radius $1/N$, $N \in \mathbb{N}(\mathcal{F})$, centered at $0 \in E$ by

$$B_{1/N} := \{X \in E \mid \|X\| \leq 1/N\}. \tag{2.1}$$

Denote $\mathcal{U} := \{B_{1/N} \mid N \in \mathbb{N}(\mathcal{F})\}$. A set $V \in E$ is a neighborhood of $X \in E$ if there is $U \in \mathcal{U}$ such that $X + U \in V$. A set $V \subset E$ is open if it is a neighborhood of all $X \in V$. The collection of open sets is a topology on E which is referred to as the topology induced by $\| \cdot \|$. The closure and interior of a set $K \subset E$ are denoted $\bar{K}$ respectively $\overset{\circ}{K}$. Unless stated otherwise, nets in E are denoted (X_N) with (by the order of almost sure dominance) directed index set $\mathbb{N}(\mathcal{F})$. A net (X_N) is Cauchy in $(E, \| \cdot \|)$ if for all $N \in \mathbb{N}(\mathcal{F})$ there is $N_0 \in \mathbb{N}(\mathcal{F})$ such that for $M \geq N_0$, $\|X_M - X_{N_0}\| \leq 1/N$. $(E, \| \cdot \|)$ is complete if every Cauchy net has a limit. Each $U \in \mathcal{U}$ is

L^0–convex: $\lambda X + (1 - \lambda)X' \in U$ for all $X, X' \in U$ and $\lambda \in L^0, 0 \leq \lambda \leq 1$,
L^0–absorbent: for all $X \in E$ there exists $Y \in L^0_{++}$ such that $X \in YU$,
L^0–balanced: $YX \in U$ for all $X \in U$ and $Y \in L^0, |Y| \leq 1$.

It is noteworthy that topologies on L^0–modules induced by L^0–convex L^0–absorbent and L^0–balanced neighborhood bases of 0 allow for convex duality, c.f. [12] for a thorough treatment.

E can be viewed as $\mathbb{R}$–vector space with scalar multiplication $\alpha X := (\alpha 1)X$ for $\alpha \in \mathbb{R}$, $1 \in L^0$ and $X \in E$. However, $(E, \| \cdot \|)$ is not a topological vector space since the map

$\mathbb{R} \to (E, \|\cdot\|)$, $\alpha \mapsto \alpha X$, is not continuous, where $0 \neq X \in E$. Indeed, $(\alpha_n) \subset \mathbb{R}$ with $\alpha_n \to 0$ does not imply $\|\alpha_n X\| = |\alpha_n 1|\|X\| \to 0$ in $(E, \|\cdot\|)$.

Nevertheless, $(E, \|\cdot\|)$ is a topological module over $(L^0, |\cdot|)$, that is, the maps

$$\begin{aligned} &(E, \|\cdot\|) \times (E, \|\cdot\|) \to (E, \|\cdot\|), (X, X') \mapsto X + X' \text{ and} \\ &(L^0, |\cdot|) \times (E, \|\cdot\|) \to (E, \|\cdot\|), (Y, X) \mapsto YX \end{aligned}$$

are continuous with respect to the product topologies. Indeed, for $X, X' \in E$, $Y \in L^0$, the properties (ii) and (iii) of definition 2.1 yield

$$(X + B_{1/2N}) + (X' + B_{1/2N}) \subset X + X' + B_{1/N}$$

$$(Y + B_{1/(3N(\|X\|+1))}) + (X + B_{1/(3N(|Y|+1))}) \subset YX + B_{1/N}$$

for all $N \in \mathbb{N}(\mathcal{F})$ with $N \geq 1/\sqrt{3}$, whence the required continuity follows.

$(E, \|\cdot\|)$ is Hausdorff. Indeed, let $X, X' \in E$ with $X \neq X'$. The neighborhood basis $\mathcal{U}$ of $0 \in E$ satisfies $\bigcap_{U \in \mathcal{U}} U = \{0\}$ so let $U \in \mathcal{U}$ such that $X - X' \notin U$. Since $(X, X') \mapsto X - X'$ is continuous, there exist $V, W \in \mathcal{U}$ such that $V - W \subset U$. Then $(X + V) \cap (X' + W) = \emptyset$, whence E is Hausdorff.

E is referred to as ordered module if it is equipped with a partial order $\geq$ that is compatible with its algebraic structure. For $X, X' \in E$ we use the notation $X \leq X'$ in place of $X' \geq X$. E is referred to as module lattice if it is an ordered module that is also a lattice. The lattice operations are denoted $X^+ := \sup\{X, 0\}$, $X^- := \sup\{-X, 0\}$, $|X| := X^+ + X^-$ so that $X = X^+ - X^-$ for all $X \in E$. Further, denote the positive convex cone $E_+ := \{X \in E \mid X \geq 0\}$. The least upper bound, if existed, of a set $C \subset E$ ($C \subset L^0$) is denoted by $\sup_{X \in C} X$ ($\text{ess.sup}_{Y \in C} Y$).

A function $\mu : E \to L^0$ is L^0–linear if $\mu(YX + Y'X') = Y\mu X + Y'\mu X'$ for all $Y, Y' \in L^0$ and $X, X' \in E$. A function $f : E \to \bar{L}^0$ is proper if $f(X) > -\infty$ for all $X \in E$ and if there is at least one $X \in E$ so that $f(X) \in L^0$. Let $f : E \to \bar{L}^0$ be a proper function. f is convex if $f(\alpha X + (1-\alpha)X') \leq \alpha f(X) + (1-\alpha) f(X')$ for all $X, X' \in E$ and $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$ (with the convention $0 \times \infty := 0$). If f is convex, then its epigraph $\text{epi} f := \{(X, Y) \in E \times L^0 \mid f(X) \leq Y\}$ is convex. f is L^0–convex if $f(YX + (1-Y)X') \leq Yf(X) + (1-Y)f(X')$ for all $X, X' \in E$ and $Y \in L^0$, $0 \leq Y \leq 1$ (again $0 \times \infty := 0$). A set $C \subset E$ is L^0–convex if $YX + (1-Y)X' \in C$ for all $X, X' \in C$ and $Y \in L^0$ with $0 \leq Y \leq 1$. If f is L^0–convex, then its epigraph is L^0–convex. f is local if $1_A f(X) = 1_A f(1_A X)$ for all $X \in E$ and $A \in \mathcal{F}$ (once more $0 \times \infty := 0$). It is well known, that if (the proper function) f is L^0–convex, then it is local , c.f. theorem 3.2 in [12]. The effective domain of f is denoted $\text{dom} f := \{X \in E \mid f(X) \in L^0\}$. f is monotone if $X \leq X'$ in the module lattice E implies $f(X) \leq f(X')$ in the almost sure sense.

Definition 2.2 *Let $(E, \|\cdot\|)$ be an L^0–normed module lattice. $\|\cdot\|$ is a lattice L^0–norm if*

(i) $\|X\| = \||X|\|$ *for all* $X \in E$,

(ii) $0 \leq X \leq X'$ *implies* $\|X\| \leq \|X'\|$ *for all* $X, X' \in E_+$.

Remark 2.3 *If* $(E, \|\cdot\|)$ *is an* L^0*–normed module lattice with lattice* L^0*–norm* $\|\cdot\|$*, then the lattice operations* $(E, \|\cdot\|) \to (E, \|\cdot\|), X \mapsto X^+, X \mapsto X^-, X \mapsto |X|$ *are uniformly continuous. Indeed, observe for instance* $|X^+ - X'^+| \leq |X - X'|$ *for all* $X, X' \in E$.

Definition 2.4 *Let* $(E, \|\cdot\|)$ *be an* L^0*–normed module. A function* $f : E \to \bar{L}^0$ *is subdifferentiable at* $X_0 \in \mathrm{dom} f$ *if there is a continuous* L^0*–linear function* $\mu : (E, \|\cdot\|) \to (L^0, |\cdot|)$ *such that*

$$\mu(X - X_0) \leq f(X) - f(X_0) \text{ for all } X \in E. \tag{2.2}$$

3 The main results

The proof of the following main results are postponed to section 5.

Theorem 3.1 *Let* $(E, \|\cdot\|)$ *be a complete* L^0*–normed module lattice with lattice* L^0*–norm* $\|\cdot\|$*. Any monotone convex local function* $f : E \to L^0$ *is continuous.*

Theorem 3.2 *Let* $(E, \|\cdot\|)$ *be a complete* L^0*–normed module lattice with lattice* L^0*–norm* $\|\cdot\|$*. Any proper monotone* L^0*–convex function* $f : E \to \bar{L}^0$ *is continuous and subdifferentiable throughout* $\overset{\circ}{\mathrm{dom}} f$.

Remark 3.3 *In fact, in theorem 3.1 and theorem 3.2 we establish a little more: let* $d : E \times E \to L^0_+$ *satisfy* (i) $d(X, X') = d(X', X)$ *for all* $X, X' \in E$*,* (ii) $d(X, X') = 0$ *if and only if* $X = X'$ *and* (iii) $d(X, Y) \leq d(X, Z) + d(Z, Y)$ *for all* $X, Y, Z \in E$*. As in* (2.1) *we can define balls* $\{X \in E \mid d(X, 0) \leq 1/N\}$*,* $N \in \mathbb{N}(\mathcal{F})$*, in turn obtain a topological module which we denote* (E, d) *and call* (E, d) *complete if every Cauchy net w.r.t.* d *has a limit. If* (E, d) *is complete, has a solid neighborhood base of* 0*, if* $1_A d(X, X') = 1_A d(1_A X, 1_A X')$ *for all* $X, X' \in E$ *and* $A \in \mathcal{F}$ *and if* (E, d) *is also a module lattice, then theorem 3.1 and theorem 3.2 remain valid on replacing* $(E, \|\cdot\|)$ *with* (E, d)*. Such* (E, d) *can be considered a module analogue of a Fréchet lattice.*

4 L^0–normed modules and financial applications

In this section we introduce important examples of complete L^0–normed module lattices. The focus is on the subsections 4.2 and 4.3. The main objective is to introduce L^p and Orlicz type L^0–modules and to present selected conditional risk measures defined on the respective modules. The first subsection 4.1 deals with the free L^0–module $(L^0)^d$, the d–fold cartesian product of L^0. This subsection does not contain any explicit financial applications. Nevertheless, it serves to illustrate the idea of how to prove completeness in our context.

Lemma 4.1 *Let* $(E, \|\cdot\|)$ *be an* L^0*–normed module. For an* L^0*–linear function* $\mu : (E, \|\cdot\|) \to (L^0, |\cdot|)$ *the following statements are equivalent:*

(i) μ *is continuous.*

(ii) μ *is continuous at* 0.

(iii) *There is* $M \in \mathbb{N}(\mathcal{F})$ *such that*

$$|\mu X| \leq M\|X\| \text{ for all } X \in E.$$

Proof. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are immediate; indeed, (iii) implies (i) as

$$|\mu X - \mu X_0| = |\mu(X - X_0)| \leq M\|X - X_0\| \text{ for all } X, X_0 \in E.$$

(ii) $\Rightarrow$ (iii) Suppose (iii) is not valid. Then, for every $M \in \mathbb{N}(\mathcal{F})$ there is $X_M \in E$ such that $|\mu X_M| > M\|X_M\|$ on a set of positive measure. Let

$$Y_M := \begin{cases} \frac{X_M}{M\|X_M\|} & \text{if } \|X_M\| \neq 0 \\ 0 & \text{else} \end{cases} .$$

Then, $|\mu Y_M| = (\mu X_M)/(M\|X_M\|) > 1$ on a set of positive measure and $\|Y_M\| \leq 1/M$ implies that $Y_M \to 0$. But this contradicts the continuity of μ at 0 as to which $|\mu Y_M| \leq 1$ for all $M \in \mathbb{N}(\mathcal{F})$ sufficiently large. □

4.1 The free L^0–module $(L^0)^d$

$E := (L^0)^d$ is a free L^0–module of rank $d \in \mathbb{N}$. The function $\|\cdot\| : (L^0)^d \to L^0_+$,

$$\|X\| := \left(\sum_{i=1}^{d} X_i^2\right)^{1/2} \quad \text{for all } X = (X_1, \ldots, X_d) \in (L^0)^d, \tag{4.3}$$

defines an L^0–norm on $(L^0)^d$. $(L^0)^d$ is finitely generated over L^0 by the elements $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, $1 \leq i \leq d$. e_i, $1 \leq i \leq d$, form a basis, that is, $\sum_{i=1}^{d} Y_i e_i = 0$ implies $Y_1, \ldots, Y_d = 0$ for all $Y_1, \ldots, Y_d \in L^0$.

Proposition 4.2 *A function* $\mu : (L^0)^d \to L^0$ *is* L^0*–linear if and only if there exists* $Z = (Z_1, \ldots Z_d) \in (L^0)^d$ *such that* $\mu X = \sum_{i=1}^{d} Z_i X_i$ *for all* $X = (X_1, \ldots, X_d) \in (L^0)^d$*. Moreover, any* L^0*–linear function* $\mu : ((L^0)^d, \|\cdot\|) \to (L^0, |\cdot|)$ *is continuous.*

Proof. Any $X = (X_1, \ldots, X_d) \in (L^0)^d$ is of the form $\sum_{i=1}^{d} X_i e_i$. By L^0–linearity

$$\mu X = \mu(\sum_{i=1}^{d} X_i e_i) = \sum_{i=1}^{d} X_i \mu e_i = \sum_{i=1}^{d} X_i Z_i, \text{ where } Z_i := \mu e_i.$$

Conversely, for $(Z_1, \ldots Z_d) \in (L^0)^d$, $(L^0)^d \to L^0, (X_1, \ldots, X_d) \mapsto \sum_{i=1}^{d} X_i Z_i$, defines an L^0–linear mapping, which is continuous, due to

$$|\mu X|^2 \leq d \max\{Z_1^2, \ldots, Z_d^2\} \sum_{i=1}^{d} X_i^2$$

for all $X = (X_1, \ldots, X_d) \in (L^0)^d$. □

Theorem 4.3 *$((L^0)^d, \|\cdot\|)$ is complete for all $d \in \mathbb{N}$.*

Proof. A net (X_N) is Cauchy (convergent) in $((L^0)^d, \|\cdot\|)$ if and only if all its components are Cauchy (convergent) in $(L^0, |\cdot|)$. Therefore, it suffices to prove $(L^0, |\cdot|)$ is complete. Let (X_N) be Cauchy in $(L^0, |\cdot|)$. Throughout, N, N_0, M, M_0 denote elements of $\mathbb{N}(\mathcal{F})$. By L^1 we denote the linear subspace of L^0 consisting of all integrable random variables.

Step 1. In this step, we prove there is $Y \in L^0_{++}$ and $X_0 \in L^1$ such that

(i) $\frac{X_N}{Y} \to X_0$ in L^1 for sufficiently large N and

(ii) $(\frac{X_N}{Y})$ is Cauchy in $(L^0, |\cdot|)$.

Since (X_N) is Cauchy in $(L^0, |\cdot|)$, there is N_0 such that

$$|X_N - X_M| \leq 1 \text{ for all } N, M \geq N_0.$$

In particular, $|X_N| \leq 1 + |X_{N_0}|$ for all $N \geq N_0$. $Y := 1 + |X_{N_0}|$ is as required. Indeed, we have $X_N/Y \in L^1$ for all $N \geq N_0$. Further, (X_N/Y) is still Cauchy in $(L^0, |\cdot|)$ which implies that (X_N/Y) is Cauchy in the Banach space L^1, whence (X_N/Y) converges to some X_0 in L^1, which proves (i) and (ii).

Step 2. In this step we prove that $X_N/Y \to X_0$ in $(L^0, |\cdot|)$ which implies that $X_N \to YX_0$ in $(L^0, |\cdot|)$. By way of contradiction, assume (X_N/Y) does not converge to X_0 in $(L^0, |\cdot|)$. Then, $\widetilde{X}_N := X_N/Y - X_0$ satisfies

$$\widetilde{X}_N \to 0 \text{ in } L^1 \text{ for sufficiently large } N \text{ and} \tag{4.4}$$

$$(\widetilde{X}_N) \text{ is Cauchy in } (L^0, |\cdot|) \text{ whereas} \tag{4.5}$$

$$(\widetilde{X}_N) \text{ does not converge to } 0 \text{ in } (L^0, |\cdot|). \tag{4.6}$$

(4.6) implies

$$\exists N_0 \forall M_0 \exists M \geq M_0 : |\widetilde{X}_M| > \frac{1}{N_0} \text{ on } A_M, \tag{4.7}$$

where $A_M \in \mathcal{F}$ with $P[A_M] > 0$. Further, from (4.5) we know that

$$\exists M_0 \forall M, N \geq M_0 : |\widetilde{X}_M - \widetilde{X}_N| \leq \frac{1}{2N_0}, \tag{4.8}$$

which together with (4.7) yields

$$\exists M \geq M_0 : |\widetilde{X}_M| > \frac{1}{N_0} \text{ on } A_M,$$

where $A_M \in \mathcal{F}$ with $P[A_M] > 0$. Since $M \geq M_0$ we know from (4.8) that

$$\forall N \geq M : |\widetilde{X}_M - \widetilde{X}_N| \leq \frac{1}{2N_0}$$

and hence

$$\forall N \geq M : |\widetilde{X}_M| - |\widetilde{X}_N| \leq \frac{1}{2N_0}.$$

From this we derive that $|\tilde{X}_N| > 1/(2N_0)$ on A_M for all $N \geq M$, in contradiction to (4.4).□

4.2 The expected shortfall risk measure on L^0–modules of L^p type

Let $(\Omega, \mathcal{E}, P)$ be a probability space with $\mathcal{F} \subset \mathcal{E}$. In this section we write $L^0(\mathcal{F}), L^0_+(\mathcal{F}), \ldots$ (instead of $L^0, L^0_+, \ldots$). The notation $L^0(\mathcal{E}), L^0_+(\mathcal{E}), \ldots$ is analogue. For $p \in [1, \infty)$ we denote $L^p(\mathcal{F}) := \{Y \in L^0(\mathcal{F}) \mid E[|Y|^p] < \infty\}$. For $p = \infty$ we denote $L^\infty(\mathcal{F})$ the space of all $\mathcal{F}$–measurable essentially bounded random variables. $L^p(\mathcal{E})$, $p \in [1, \infty]$, is given analogue.

We recall that the classical conditional expectation $E[\cdot \mid \mathcal{F}] : L^1(\mathcal{E}) \to L^1(\mathcal{F})$ extends to the (generalized) conditional expectation $E[\cdot \mid \mathcal{F}] : L^0_+(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by

$$E[X \mid \mathcal{F}] := \lim_{n \to \infty} E[X \wedge n \mid \mathcal{F}]. \tag{4.9}$$

Beppo–Levi's monotone convergence theorem yields for all $X, X' \in L^0_+(\mathcal{E})$ and $Y \in L^0_+(\mathcal{F})$

(i) $YE[X \mid \mathcal{F}] = E[YX \mid \mathcal{F}]$,

(ii) $E[X + X' \mid \mathcal{F}] = E[X \mid \mathcal{F}] + E[X' \mid \mathcal{F}]$,

(iii) $E[X] = E[E[X \mid \mathcal{F}]]$,

where the last expectation might equal ∞, however, as such, it remains well defined. For $p \in [1, \infty]$ we define the function $\|\cdot\|_p : L^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by

$$\|X\|_p := \begin{cases} E[|X|^p \mid \mathcal{F}]^{1/p} & \text{if } p \in [1, \infty) \\ \text{ess.inf}\{Y \in \bar{L}^0_+(\mathcal{F}) \mid Y \geq |X|\} & \text{if } p = \infty \end{cases}, \tag{4.10}$$

and

$$L^p_{\mathcal{F}}(\mathcal{E}) := \{X \mid X \in L^0(\mathcal{E}), \|X\|_p \in L^0(\mathcal{F})\}.$$

If equipped with the order of almost sure dominance, the properties of the (generalized) conditional expectation guarantee that $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$ is an L^0–normed module lattice with lattice norm $\|\cdot\|_p$.

Proposition 4.4 *$L^p_{\mathcal{F}}(\mathcal{E})$ has the product structure*

$$L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F})L^p(\mathcal{E}) := \{YX \mid Y \in L^0(\mathcal{F}),\ X \in L^p(\mathcal{E})\}. \tag{4.11}$$

Proof. $L^p(\mathcal{E})$ is a subspace of $L^p_{\mathcal{F}}(\mathcal{E})$. Hence, $\supset$ in (4.11) follows from property (ii) in definition 2.1 of $\|\cdot\|_p$. The reverse inclusion follows as every $X \in L^p_{\mathcal{F}}(\mathcal{E})$ is of the form

$$X = (1 + \|X\|_p) \frac{X}{1 + \|X\|_p} \in L^0(\mathcal{F})L^p(\mathcal{E}).$$

□

(4.11) suggests an alternative view of $L^p_{\mathcal{F}}(\mathcal{E})$, namely $L^p_{\mathcal{F}}(\mathcal{E})$ is the smallest $L^0(\mathcal{F})$–module containing the classical $L^p(\mathcal{E})$ space. That is, $L^p(\mathcal{E})$ generates $L^p_{\mathcal{F}}(\mathcal{E})$.

We work with the convention that the (generalized) conditional expectation $E[\cdot \mid \mathcal{F}] : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is understood as

$$E[X \mid \mathcal{F}] := E[X^+ \mid \mathcal{F}] - E[X^- \mid \mathcal{F}], \tag{4.12}$$

the right hand side of which is understood as in (4.9). Note that for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$, both $E[X^+ \mid \mathcal{F}]$ and $E[X^- \mid \mathcal{F}]$ are in $L^0(\mathcal{F})$ and consequently $E[X \mid \mathcal{F}] \in L^0(\mathcal{F})$. Further, note that $E[\cdot \mid \mathcal{F}] : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is $L^0(\mathcal{F})$–linear.

For $p \in (1, \infty)$ let $q := p/(p-1)$, if $p = 1$ let $q := \infty$ and if $p = \infty$ let $q := 1$. Then, for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $X' \in L^q_{\mathcal{F}}(\mathcal{E})$

$$\|XX'\|_1 \leq \|X\|_p \|X'\|_q. \tag{4.13}$$

Indeed, with the classical Hölder inequality for conditional expectations we know that

$$\|(X \wedge n)(X' \wedge n)\|_1 \leq \|X \wedge n\|_p \|X' \wedge n\|_q$$

for all $n \in \mathbb{N}$ and Beppo–Levi's monotone convergence theorem yields the assertion. Monotone convergence shows that (4.13) even holds for $p \in \bar{L}^0_+(\mathcal{F})$, $p \geq 1$, with $q := p/(p-1)$ with the conventions $q := 1$ and $q := \infty$ if $p = \infty$ and $p = 1$, respectively.

Theorem 4.5 *Let $p \in [1, \infty)$ and $q := p/(p-1)$ with $q := \infty$ for $p = 1$. Every continuous $L^0(\mathcal{F})$–linear function $\mu : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ is of the form*

$$\mu X = E[ZX \mid \mathcal{F}] \tag{4.14}$$

for some $Z \in L^q_{\mathcal{F}}(\mathcal{E})$. Conversely, every function $\mu : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ of the form (4.14) is continuous $L^0(\mathcal{F})$–linear.

Proof. For $Z \in L^q_{\mathcal{F}}(\mathcal{E})$, $E[Z\cdot \mid \mathcal{F}] : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ is $L^0(\mathcal{F})$–linear and (4.13) guarantees that $E[ZX \mid \mathcal{F}] \in L^0(\mathcal{F})$ for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and the continuity.

Conversely, let $\mu : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ be continuous $L^0(\mathcal{F})$–linear. From lemma 4.1 we know there is $M \in \mathbb{N}(\mathcal{F})$ such that

$$|\mu X| \leq M \|X\|_p \text{ for all } X \in L^p_{\mathcal{F}}(\mathcal{E})$$

and hence we can define

$$\widetilde{\mu} X := \frac{\mu X}{M} \text{ for all } X \in L^p_{\mathcal{F}}(\mathcal{E}).$$

Then, for all $X' \in L^p(\mathcal{E})$, we have

$$E[|\widetilde{\mu} X'|^p] \leq E[E[|X'|^p \mid \mathcal{F}]].$$

Jensen's inequality further implies

$$|E[\widetilde{\mu} X']| \leq E[|X'|^p]^{1/p}$$

which means that $E \circ \widetilde{\mu} : L^p(\mathcal{E}) \to \mathbb{R}$ is continuous. The topological dual of $L^p(\mathcal{E})$ can be identified with $L^q(\mathcal{E})$. Hence, there is $Z' \in L^q(\mathcal{E})$ such that

$$E[\widetilde{\mu}X'] = E[Z'X'] \text{ for all } X' \in L^p(\mathcal{E}).$$

From $L^0(\mathcal{F})$–linearity of $\widetilde{\mu}$ we derive

$$\widetilde{\mu}X' = E[Z'X' \mid \mathcal{F}] \text{ for all } X' \in L^p(\mathcal{E})$$

which is equivalent to

$$\mu X' = E[Z'MX' \mid \mathcal{F}] \text{ for all } X' \in L^p(\mathcal{E}).$$

(4.11) guarantees that $Z := Z'M \in L^q_{\mathcal{F}}(\mathcal{E})$ and that every $X \in L^p_{\mathcal{F}}(\mathcal{E})$ is of the form $X = YX'$ for some $Y \in L^0(\mathcal{F})$, $X' \in L^p(\mathcal{E})$. From this we conclude

$$\mu X = Y\mu X' = YE[ZX' \mid \mathcal{F}] = E[ZX \mid \mathcal{F}].$$

□

Theorem 4.6 *$(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|)$ is complete for all $p \in [1, \infty]$.*

Proof. The proof is similar to that of theorem 4.3. We sketch the main steps. Throughout, N, N_0, M, M_0 denote elements of $\mathbb{N}(\mathcal{F})$. Let (X_N) be Cauchy in $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$ for some $p \in [1, \infty]$. Since (X_N) is Cauchy there is N_0 such that $\|X_N\|_p \leq 1 + \|X_{N_0}\|_p =: Y$ for all $N \geq N_0$. We derive $X_N/Y \in L^p(\mathcal{E})$ for all $N \geq N_0$,

(i) $\frac{X_N}{Y} \to X_0$ in $L^p(\mathcal{E})$ for sufficiently large N and

(ii) $\left(\frac{X_N}{Y}\right)$ is Cauchy in $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$.

We prove (i) and (ii) in case $p \in [1, \infty)$. The case of $p = \infty$ is analogue, only the notation is different. For every N_0 there is M_0 such that

$$\forall N, M \geq M_0 : \|\frac{X_N}{Y} - \frac{X_M}{Y}\|_p \leq \frac{1}{N_0}$$

which implies that

$$\forall N, M \geq M_0 : E\left[|\frac{X_N}{Y} - \frac{X_M}{Y}|^p\right]^{1/p} \leq E\left[\left(\frac{1}{N_0}\right)^p\right]^{1/p}.$$

Hence (X_N/Y) is Cauchy in the Banach space $L^p(\mathcal{E})$ and converges to some X_0 in $L^p(\mathcal{E})$, which proves (i) and (ii).

In a second step one shows by way of contradiction that (X_N/Y) converges to X_0 in $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|)$. The proof is identical to the respective part of the proof of theorem 4.3. □

Let $\lambda \in L^0(\mathcal{F})$, $0 < \lambda < 1$. The conditional expected shortfall $\rho : L^1_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ at level λ is

$$\rho(X) := \operatorname*{ess.sup}_{Z \in \mathcal{P}} E[ZX \mid \mathcal{F}], \tag{4.15}$$

where

$$\mathcal{P} := \left\{ Z \in L^0(\mathcal{F}) \mid Z \leq 0, |Z| \leq \frac{1}{\lambda}, E[Z \mid \mathcal{F}] = -1 \right\}.$$

The conditional expected shortfall is antitone ($\rho(X) \leq \rho(X')$ for all $X, X' \in L^1_{\mathcal{F}}(\mathcal{E})$ with $X \geq X'$) and convex local. By theorem 3.1 (note that $\widetilde{\rho}(\cdot) := \rho(-\cdot)$ satisfies the respective assumptions), ρ is continuous. Further, by theorem 3.2, $\partial\rho(X) \neq \emptyset$ for all $X \in L^1_{\mathcal{F}}(\mathcal{E})$.

Lemma 4.7 *Let $Y_0 \in L^0(\mathcal{F})$ and $f : (L^0(\mathcal{F}), |\cdot|) \to (L^0(\mathcal{F}), |\cdot|)$ a continuous local function. If there are $Y_1, Y_2 \in L^0(\mathcal{F})$ with $f(Y_1) < Y_0 < f(Y_2)$, then there is $Y^* \in L^0(\mathcal{F})$ with $f(Y^*) = Y_0$.*

Proof. Define

$$Y^* := \operatorname{ess.inf}\{Y \in L^0(\mathcal{F}) \mid f(Y) \geq Y_0, Y \geq Y_1\}.$$

Then $Y^* \in L^0(\mathcal{F})$ and we claim that Y^* is as required. Indeed, assume by way of contradiction that $f(Y^*) > Y_0$ with positive probability and let $Y_N := (Y^* - 1/N) \vee Y_1$, $N \in \mathbb{N}(\mathcal{F})$. Then $Y_N \to Y^*$. Further, $f(Y_N) < Y_0$; else, since f is local, we would derive a contradiction to the minimality of Y^*. Altogether, we derive $f(Y_N) < Y_0 < f(Y^*)$ with positive probability in contradiction to the continuity of f as to which $Y_N \to Y^*$ implies $f(Y_N) \to f(Y^*)$. ☐

Definition 4.8 *A random variable $X \in L^0(\mathcal{E})$ is conditionally continuously distributed if the map*

$$(L^0, |\cdot|) \to (L^0, |\cdot|), Y \mapsto E[1_{\{X \leq Y\}} \mid \mathcal{F}],$$

is continuous.

If $X \in L^0(\mathcal{E})$ is conditionally continuously distributed, then lemma 4.7 implies that there is $Y^* \in L^0(\mathcal{F})$ with $E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = \lambda$. Indeed, since

$$\lambda < 1 = \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F})} E[1_{\{X \leq Y\}} \mid \mathcal{F}]$$
$$\lambda > 0 = \operatorname*{ess.inf}_{Y \in L^0(\mathcal{F})} E[1_{\{X \leq Y\}} \mid \mathcal{F}]$$

there are nets $(Y_N^-), (Y_N^+)$ with

$$E[1_{\{X \leq Y_N^-\}} \mid \mathcal{F}] < \lambda < E[1_{\{X \leq Y_N^+\}} \mid \mathcal{F}]$$

for sufficiently large $N \in \mathbb{N}(\mathcal{F})$ and the assumptions of lemma 4.7 are met. Such Y^* can be viewed as a conditional λ–quantile of X. For a detailed discussion of such quantiles we refer to [27].

Proposition 4.9 *For all conditionally continuously distributed $X \in L^1_{\mathcal{F}}(\mathcal{E})$ and $\lambda \in L^0(\mathcal{F})$ with $0 < \lambda < 1$,*

$$-\frac{1}{\lambda} 1_{\{X \leq Y^*\}} \in \partial \rho(X),$$

where $Y^* \in L^0(\mathcal{F})$ is such that $E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = \lambda$.

Proof. Let $Y^* \in L^0(\mathcal{F})$ with $E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = \lambda$ and $Z^* := -(1/\lambda) 1_{\{X \leq Y^*\}}$. Then,

$$E[Z^* \mid \mathcal{F}] = -(1/\lambda) E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = -1$$

and hence $Z \in \mathcal{P}$. Further, for all $Z \in \mathcal{P}$ we have $E[(Z^* - Z) Y^*] = 0$. Thus,

$$\begin{aligned} E[(Z^* - Z) X \mid \mathcal{F}] &= E[\underbrace{1_{\{X \leq Y^*\}}(Z^* - Z)(X - Y^*)}_{\geq 0} \mid \mathcal{F}] \\ &\quad + E[\underbrace{1_{\{X > Y^*\}}(Z^* - Z)(X - Y^*)}_{\geq 0} \mid \mathcal{F}] \geq 0. \end{aligned}$$

Hence, Z^* optimizes (4.15) and is therefore a subgradient of ρ at X. □

4.3 The entropic risk measure on L^0–modules of Orlicz type

Throughout the most recent literature, there has been some attention on risk measures defined on Orlicz spaces [7, 8, 2]. Orlicz spaces and Orlicz hearts share some useful properties with L^p spaces, for instance, they are Banach spaces and admit nice duality. However, as they are technically more challenging it may sometimes not be clear why to prefer them over the familiar L^p spaces.

There is, nevertheless, at least one reason to employ Orlicz space theory in the context of risk measures. In the likely case of an empty interior of the effective domain of a risk measure on L^p, one can only hope to establish Fenchel–Moreau type dual representations while subdifferentiability will certainly be out of reach. However, if the L^p space topology is replaced by a finer Orlicz space topology the interior of the effective domain may no longer be empty. For example, the static entropic risk measure is lower semi continuous on L^1, Fenchel–Moreau's theorem therefore applies but the interior of its effective domain is empty. On the contrary, if considered on a suitable Orlicz space, the interior of its effective domain is non empty and throughout the entropic risk measure becomes subdifferentiable.

The aim of this section is to present similar results in the context of conditional risk measures on L^0–modules. To this end, we shall adopt the setup and notation of the previous section. Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing convex function with $\varphi(0) = 0$. The respective Orlicz space is

$$L^\varphi(\mathcal{E}) := \{X \in L^0(\mathcal{E}) \mid \exists \lambda \in (0, \infty) : \ E[\varphi(|X/\lambda|)] < \infty\}.$$

Recall that $L^\varphi(\mathcal{E})$ endowed with the Luxemburg–norm $\| \cdot \| : L^\varphi(\mathcal{E}) \to \mathbb{R}_+$,

$$\|X\| := \inf\{\lambda \in (0, \infty) \mid E[\varphi(|X/\lambda|)] \leq 1\}$$

is a Banach space. We define a module analogue of the Orlicz space by

$$L^{\varphi}_{\mathcal{F}}(\mathcal{E}) := \{X \in L^0(\mathcal{E}) \mid \exists Y \in L^0_{++}(\mathcal{F}) : E[\varphi(|X/Y|) \mid \mathcal{F}] \in L^0(\mathcal{F})\}$$

and the map $\|\cdot\|_{\varphi} : L^0(\mathcal{E}) \to \bar{L}^0_{+}(\mathcal{F})$,

$$\|X\|_{\varphi} := \text{ess.inf}\{Y \in L^0_{++}(\mathcal{F}) \mid E[\varphi(|X/Y|) \mid \mathcal{F}] \leq 1\}$$

which generalizes the Luxemburg–norm. Then

$$L^{\varphi}_{\mathcal{F}}(\mathcal{E}) = \{X \in L^0(\mathcal{E}) \mid \|X\|_{\varphi} \in L^0(\mathcal{F})\}. \tag{4.16}$$

Indeed, the inclusion $\supset$ in (4.16) follows by definition. As to the reverse, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0_{++}(\mathcal{F})$ with $E[\varphi(|X/Y|) \mid \mathcal{F}] \in L^0(\mathcal{F})$. Then there is $Y' \in L^0(\mathcal{F})$, $Y' \geq 1$, such that $E[\varphi(|X/Y|) \mid \mathcal{F}]/Y' \leq 1$. By convexity of φ,

$$E[\varphi(|X/(YY')|) \mid \mathcal{F}] \leq E[\varphi(|X/Y|) \mid \mathcal{F}]/Y' \leq 1$$

so that $\subset$ in (4.16) follows.

Proposition 4.10 *$(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$ is an L^0–normed module lattice with lattice L^0–norm $\|\cdot\|_{\varphi}$.*

Proof. First, we establish that $\|\cdot\|_{\varphi}$ is an L^0–norm. To this end, observe that (4.16) implies $\|\cdot\|_{\varphi}$ maps $L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ into $L^0_{+}(\mathcal{F})$.

To prove (i) of definition 2.1, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and suppose $\|X\|_{\varphi} = 0$. Convexity of $\varphi(|\cdot|)$ yields

$$nE[\varphi(|X|) \mid \mathcal{F}] \leq E[\varphi(n|X|) \mid \mathcal{F}] \leq 1 \quad \text{for all } n \in \mathbb{N}$$

which implies $\varphi(|X|) = 0$, whence $X = 0$.

To prove (ii) of definition 2.1, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E}), Y' \in L^0(\mathcal{F})$. Then

$$\begin{aligned}
\|Y'X\|_{\varphi} &= \text{ess.inf}\{Y \in L^0_{++}(\mathcal{F}) \mid E[\varphi(|Y'X/Y|) \mid \mathcal{F}] \leq 1\} \\
&= 1_{\{|Y'|>0\}}\text{ess.inf}\{|Y'|Y \mid Y \in L^0_{++}(\mathcal{F}), E[\varphi(|Y'X/(Y'Y)|) \mid \mathcal{F}] \leq 1\} \\
&= |Y'|\|X\|_{\varphi}.
\end{aligned}$$

To prove (iii) of definition 2.1 let $X_1, X_2 \in L^{\varphi}_{\mathcal{F}}(\mathcal{E}), Y_1, Y_2 \in L^0_{++}(\mathcal{F})$ with $E[\varphi(|X_1/Y_1|) \mid \mathcal{F}] \leq 1$ and $E[\varphi(|X_2/Y_2|) \mid \mathcal{F}] \leq 1$. Convexity of $\varphi(|\cdot|)$ yields

$$E\left[\varphi\left(|\frac{Y_1}{Y_1+Y_2}\frac{X_1}{Y_1} + \frac{Y_2}{Y_1+Y_2}\frac{X_2}{Y_2}|\right) \mid \mathcal{F}\right] \leq 1,$$

whence $\|X_1 + X_2\|_{\varphi} \leq Y_1 + Y_2$.

By definition, $\|\cdot\|_{\varphi}$ is a lattice L^0–norm. Further, (4.16) and properties (ii) and (iii) of $\|\cdot\|_{\varphi}$ imply that $L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ is an L^0–module. □

Proposition 4.11 *$L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ has the product structure $L^{\varphi}_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F})L^{\varphi}(\mathcal{E})$.*

Proof. To show that $L^{\varphi}_{\mathcal{F}}(\mathcal{E}) \subset L^0(\mathcal{F})L^{\varphi}(\mathcal{E})$, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0_{++}$ such that $E[\varphi(X/Y) \mid \mathcal{F}] \in L^0(\mathcal{F})$. By convexity of $\varphi(|\cdot|)$,

$$E\left[\varphi\left(|\frac{X/Y}{1+E[\varphi(|X/Y|) \mid \mathcal{F}]}|\right)\right] \leq E\left[\frac{E[\varphi(|X/Y|) \mid \mathcal{F}]}{1+E[\varphi(|X/Y|) \mid \mathcal{F}]}\right] \leq 1. \tag{4.17}$$

This implies

$$X = (1+E[\varphi(|X/Y|) \mid \mathcal{F}])Y\frac{X/Y}{1+E[\varphi(|X/Y|) \mid \mathcal{F}]} \in L^0(\mathcal{F})L^{\varphi}(\mathcal{E}).$$

To establish the reverse inclusion, let $X' \in L^{\varphi}(\mathcal{E})$, $Y \in L^0(\mathcal{F})$ and $X := YX'$. By definition, there is $\lambda \in (0,\infty)$ such that $E[\varphi(|X'/\lambda|)] \leq 1$. Hence, $E[\varphi(|X'/\lambda|) \mid \mathcal{F}] \in L^0_+(\mathcal{F})$ and therefore by convexity of φ

$$E\left[\varphi\left(|\frac{X'/\lambda}{1+E[\varphi(|X'/\lambda|)]}|\right) \mid \mathcal{F}\right] \leq \frac{E[\varphi(|X'/\lambda|) \mid \mathcal{F}]}{1+E[\varphi(|X'/\lambda|] \mid \mathcal{F}]} \leq 1.$$

This shows

$$\frac{1}{\lambda(1+E[\varphi(|X/\lambda|) \mid \mathcal{F}])}X' \in L^{\varphi}_{\mathcal{F}}(\mathcal{E}).$$

Since $L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ is an L^0–module we derive $X' \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and in turn $YX' = X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$. □

Theorem 4.12 *$(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$ is complete.*

Proof. Again, the proof is similar to that of theorem 4.3 and we sketch the main steps. Throughout, N, N_0, M, M_0 denote elements of $\mathbb{N}(\mathcal{F})$. Let (X_N) be Cauchy in $(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$. Since (X_N) is Cauchy there is N_0 such that $\|X_N\|_{\varphi} \leq 1 + \|X_{N_0}\|_{\varphi} =: Y$ for all $N \geq N_0$. (4.17) yields $X_N/Y \in L^{\varphi}(\mathcal{E})$ for all $N \geq N_0$. Moreover,

(i) $\frac{X_N}{Y} \to X_0$ in $L^{\varphi}(\mathcal{E})$ for sufficiently large N and

(ii) $\left(\frac{X_N}{Y}\right)$ is Cauchy in $(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$.

Indeed, for every $N_0 \in \mathbb{N}(\mathcal{F})$ (in particular for $N \equiv n$, $n \in \mathbb{N}$) there is M_0 such that

$$\forall N, M \geq M_0 : \|\frac{X_N}{Y} - \frac{X_M}{Y}\|_{\varphi} \leq \frac{1}{N_0}.$$

Hence, by definition of $\|\cdot\|_{\varphi}$,

$$\forall N, M \geq M_0 : E\left[\varphi\left(|\frac{X_N}{Y} - \frac{X_M}{Y}|N_0\right)\right] \leq 1$$

Thus, (X_N/Y) is Cauchy in the Banach space $L^{\varphi}(\mathcal{E})$ and converges to some $X_0 \in L^{\varphi}(\mathcal{E})$, which proves (i) and (ii).

Again, by way of contradiction one shows that (X_N/Y) converges to X_0 in $(L^\varphi_\mathcal{F}(\mathcal{E}), \|\cdot\|_\varphi)$. The proof is identical to the respective part of the proof of theorem 4.3. □

Let $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(x) := \exp(x) - 1$, and define the entropic risk measure $\rho : L^\varphi_\mathcal{F}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ with risk aversion coefficient $\gamma \in (0,\infty)$ as

$$\rho(X) := \frac{1}{\gamma} \log E[\exp(-\gamma X) \mid \mathcal{F}].$$

It is shown in [12] that ρ admits Fenchel–Moreau type dual representations if considered as a function on $L^p_\mathcal{F}(\mathcal{E})$, $p \in [1,\infty]$. However, to establish subdifferentiability results, we consider ρ as a function on the Orlicz type module $L^\varphi_\mathcal{F}(\mathcal{E})$ and define

$$\chi := \{X \in L^\varphi_\mathcal{F}(\mathcal{E}) \mid \exists \varepsilon \in L^0_{++}(\mathcal{F}) : E[\exp((1+\varepsilon)\gamma X^-) \mid \mathcal{F}] \in L^0(\mathcal{F})\}.$$

The following lemma is a module variant of lemma 32 in [2].

Lemma 4.13 $\chi \subset \overset{\circ}{\operatorname{dom}}\,\rho$.

Proof. Let $X \in \chi$ with $E[\exp((1+\varepsilon)\gamma X^-) \mid \mathcal{F}] \in L^0(\mathcal{F})$ for some $\varepsilon \in L^0_{++}(\mathcal{F})$. Let $p := (1+\varepsilon)/(1+\varepsilon/2)$ with conjugate $q := p/(p-1) = 2(1+\varepsilon)/\varepsilon$. Since there is $N \in \mathbb{N}(\mathcal{F})$ with $1/N \leq 1/((1+\varepsilon/2)\gamma q)$ it suffices to show that $X + B_{1/((1+\varepsilon/2)\gamma q)} \subset \chi$, where $B_{1/((1+\varepsilon/2)\gamma q)} := \{Y \in L^\varphi_\mathcal{F}(\mathcal{E}) \mid \|Y\|_\varphi \leq 1/((1+\varepsilon/2)\gamma q)\}$.

To this end, fix $Y \in B_{1/((1+\varepsilon/2)\gamma q)}$. By definition,

$$E[\exp((1+\varepsilon/2)\gamma q|Y|) \mid \mathcal{F}] \leq E[\exp(|Y|/\|Y\|_\varphi) \mid \mathcal{F}] \leq 1.$$

Hence, Hölder's inequality yields

$$\begin{aligned}
& E[\exp((1+\varepsilon/2)\gamma(X+Y)^-) \mid \mathcal{F}] \\
\leq\ & E[\exp((1+\varepsilon/2)\gamma X^-)\exp((1+\varepsilon/2)\gamma Y^-) \mid \mathcal{F}] \\
\leq\ & E[\exp((1+\varepsilon/2)\gamma X^-)^p \mid \mathcal{F}]^{1/p} E[\exp((1+\varepsilon/2)\gamma Y^-)^q \mid \mathcal{F}]^{1/q} \\
\leq\ & E[\exp((1+\varepsilon)\gamma X^-) \mid \mathcal{F}]^{1/p} E[\exp((1+\varepsilon/2)\gamma q|Y|) \mid \mathcal{F}]^{1/q} \\
\leq\ & E[\exp((1+\varepsilon)\gamma X^-) \mid \mathcal{F}]^{1/p} \in L^0(\mathcal{F}).
\end{aligned}$$

This shows that χ is open in $L^\varphi_\mathcal{F}(\mathcal{E})$. Moreover, for all $X \in \chi$,

$$E[\exp(-\gamma X) \mid \mathcal{F}] \leq 1 + E[\exp(\gamma X^-) \mid \mathcal{F}] \in L^0(\mathcal{F}).$$

Hence, $\chi \subset \operatorname{dom}\rho$. We conclude $\chi \subset \overset{\circ}{\chi} \subset \overset{\circ}{\operatorname{dom}}\rho$. □

The entropic risk measure ρ is proper antitone L^0–convex and hence continuous and subdifferentiable throughout $\overset{\circ}{\operatorname{dom}}\rho$, in particular throughout χ, by theorem 3.2.

Lemma 4.14 *Let $X_0 \in \chi$ and $Z_0 := \exp(-\gamma X_0)/E[\exp(-\gamma X_0) \mid \mathcal{F}]$. Then, $\mu : L^\varphi_\mathcal{F}(\mathcal{E}) \to L^0(\mathcal{F})$, $\mu(X) := E[(-Z_0)X \mid \mathcal{F}]$, defines a continuous L^0–linear function.*

Proof. First, we show that μ maps $L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ into $L^0(\mathcal{F})$. To this end, let $\varepsilon \in L^0_{++}(\mathcal{F})$ such that $E[\exp(-\gamma(1+\varepsilon)X_0^-) \mid \mathcal{F}] \in L^0(\mathcal{F})$ which exists since $X_0 \in \chi$. Define $p := 1+\varepsilon$ with conjugate $q := (1+\varepsilon)/\varepsilon$. There exists $c \in \mathbb{N}(\mathcal{F})$ such that $X^q \leq c(1+\exp(X))$ for all $X \in L^0_+(\mathcal{E})$ showing that $L^{\varphi}_{\mathcal{F}}(\mathcal{E}) \subset L^q_{\mathcal{F}}(\mathcal{E})$. Hence, Hölder's inequality yields for all $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$

$$|\mu(X)| \leq (1 + E[\exp(-\gamma(1+\varepsilon)X_0^-) \mid \mathcal{F}]^{1/p}) E[|X|^q \mid \mathcal{F}]^{1/q} \in L^0(\mathcal{F}).$$

Further, μ is antitone L^0–linear by definition and hence, in particular, antitone convex local. Thus, μ is continuous L^0–linear by theorem 3.1. □

Proposition 4.15 *Let X_0, Z_0 and μ be as in lemma 4.14. Then, $\mu \in \partial\rho(X_0)$.*

Proof. On adapting the proof of lemma 3.29 in [16], we derive for all $X \in \operatorname{dom}\rho$

$$\begin{aligned}\frac{1}{\gamma}\log E[\exp(-\gamma X) \mid \mathcal{F}] &\geq E[-Z_0 X \mid \mathcal{F}] - \frac{1}{\gamma}E[Z_0 \log Z_0 \mid \mathcal{F}] \\ \frac{1}{\gamma}\log E[\exp(-\gamma X_0) \mid \mathcal{F}] &= E[-Z_0 X_0 \mid \mathcal{F}] - \frac{1}{\gamma}E[Z_0 \log Z_0 \mid \mathcal{F}].\end{aligned}$$

Hence, for any $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$

$$\frac{1}{\gamma}\log E[\exp(-\gamma X) \mid \mathcal{F}] - \frac{1}{\gamma}\log E[\exp(-\gamma X_0) \mid \mathcal{F}] \geq E[(-Z_0)(X - X_0) \mid \mathcal{F}].$$

Thus, $\mu \in \partial\rho(X_0)$. □

5 Proof of the main results

5.1 Proof of theorem 3.1

Let E be an L^0–module and (X_N) be a net in E. (X_N) is directed if

$$1_A X_N + 1_{A^c} X_{N'} = X_{1_A N + 1_{A^c} N'} \tag{5.18}$$

for all $N, N' \in \mathbb{N}(\mathcal{F})$ and $A \in \mathcal{F}$. If (X_N) is directed so is every subnet. Denote (X_n) the sequence obtained via $X_n := X_N$ for the constant map $N \equiv n$, $n \in \mathbb{N}$. With (X_n) we define a net $(\widetilde{X}_N)$ by

$$\widetilde{X}_N(\omega) := X_{N(\omega)}(\omega) \text{ for all } \omega \in \Omega \text{ and } N \in \mathbb{N}(\mathcal{F}). \tag{5.19}$$

If (X_N) is directed then it is recovered by (5.19), that is, $X_N = \widetilde{X}_N$ for all $N \in \mathbb{N}(\mathcal{F})$. In this sense, directed nets correspond to sequences.

For a set $C \subset E$ we define the map $M(\cdot \mid C) : E \to \mathcal{F}$,

$$M(X \mid C) := \operatorname{ess.sup}\{A \in \mathcal{F} \mid 1_A X \in C\}. \tag{5.20}$$

Lemma 5.1 *Let $(E, \|\cdot\|)$ be an L^0–normed module, $f : E \to \bar{L}^0$ a local function and $X_0 \in$ dom$f$. Equivalent are:*

(i) *f is continuous w.r.t. $\|\cdot\|$ and $|\cdot|$ at X_0.*

(ii) *$X_N \to X_0$ implies $f(X_N) \to f(X_0)$ for all directed nets (X_N) in E.*

Proof. We only have to prove that (ii) implies (i). On replacing f by $f(\cdot + X_0) - f(X_0)$ we may assume that $X_0 = f(X_0) = 0$ (leaving hypotheses and conclusion unchanged). Let $N_0 \in \mathbb{N}(\mathcal{F})$. We have to show that $f^{-1}(B_{1/N_0})$ is a neighborhood of $0 \in E$. Define

$$\varepsilon^* := \text{ess.sup}\{\varepsilon \in L^0_+ \mid B_\varepsilon \subset f^{-1}(B_{1/N_0})\} \wedge 1,$$

where

$$B_\varepsilon := \{X \in E \mid \|X\| \leq \varepsilon\}.$$

It suffices to show that $\varepsilon^* \in L^0_{++}$. Indeed, since f is local, $\mathcal{G} := \{\varepsilon \in L^0_+ \mid B_\varepsilon \subset f^{-1}(B_{1/N_0})\}$ is directed upwards. Hence, there is an increasing sequence $(\varepsilon_n) \subset \mathcal{G}$ with $\varepsilon_n \nearrow \varepsilon^*$ a.s. Thus, $A_1 := \{\varepsilon_1 > 0\}$, $A_n := \{\varepsilon_n > 0\} \setminus A_{n-1}$, $n \geq 2$, satisfies $\bigcup_{n\in\mathbb{N}} A_n = \Omega$ as $\varepsilon^* > 0$. For $N \in \mathbb{N}(\mathcal{F})$ such that

$$\frac{1}{N} \leq \sum_{n\in\mathbb{N}} 1_{A_n}\varepsilon_n$$

we see that $B_{1/N} \subset f^{-1}(B_{1/N_0})$ since f is local. Thus, f is continuous at $0 \in E$ if $\varepsilon^* > 0$.

By way of contradiction let us assume that $P[A] > 0$, where $A := \{\varepsilon^* = 0\}$. Fix $n_0 \in \mathbb{N}$. We will show that there is $\bar{X}_{n_0} \in E$ with $\bar{X}_{n_0} \in B_{1/n_0}$ and

$$1_B\bar{X}_{n_0} \notin 1_B f^{-1}(B_{1/N_0}) \text{ for all } B \in \mathcal{F} \text{ with } B \subset A \text{ and } P[B] > 0. \tag{5.21}$$

To this end, consider the collection $\mathcal{D}$ of all sets $B \in \mathcal{F}$ such that there is $X \in E$ with $X \in B_{1/n_0}$ and

$$1_{B'}X \notin 1_{B'} f^{-1}(B_{1/N_0}) \text{ for all } B' \in \mathcal{F} \text{ with } B' \subset B \text{ and } P[B'] > 0. \tag{5.22}$$

We claim that $A \subset \text{ess.sup}\mathcal{D}$. Indeed, assume that $P[C] > 0$, $C := A \setminus \text{ess.sup}\mathcal{D}$. Since $C \subset A$ and in turn $\varepsilon^* = 0$ on C we derive from the maximality of ε^* that $1_C B_{1/n_0}$ cannot be a subset of $1_C f^{-1}(B_{1/N_0})$. Hence, take $Z \in 1_C B_{1/n_0} \setminus 1_C f^{-1}(B_{1/N_0})$ and observe that $P[C'] > 0$, $C' := C \setminus M(Z \mid f^{-1}(B_{1/N_0}))$, since f is local. But this contradicts the maximality of ess.sup$\mathcal{D}$ since

$$1_{B'}Z \notin 1_{B'} f^{-1}(B_{1/N_0}) \text{ for all } B' \in \mathcal{F} \text{ with } B' \subset C' \text{ and } P[B'] > 0.$$

Thus, $A \subset \text{ess.sup}\mathcal{D}$. Since the collection $\mathcal{D}$ is directed upwards there exists an increasing sequence (C_n) in $\mathcal{D}$ with $C_n \nearrow \text{ess.sup}\mathcal{D}$ and a corresponding sequence (X_n) satisfying (5.22). Then

$$\bar{X}_{n_0} := \sum_{n\in\mathbb{N}} 1_{C_n} X_n$$

is an element of B_{1/n_0} and satisfies (5.21), that is, $\bar{X}_{n_0}$ is as required.

Finally, we proceed in the same manner as in (5.19) and construct a directed net $(\bar{X}_N)$ by means of the sequence $(\bar{X}_n)$ we just constructed and we observe that

$$\begin{aligned}&\bar{X}_N \to 0 \text{ but still}\\ &1_B f(X_N) \notin 1_B B_{1/N_0}(Y)\end{aligned}$$

for all $N \in \mathbb{N}(\mathcal{F})$ and $B \in \mathcal{F}$ with $B \subset A$ and $P[B] > 0$ which contradicts (ii) and concludes the proof. □

The next lemma is a module variant of theorem 8.43 in [1].

Lemma 5.2 *Let $(E, \|\cdot\|)$ be an L^0–normed module lattice with lattice L^0–norm $\|\cdot\|$.*

- (i) *E_+ is closed in $(E, \|\cdot\|)$.*
- (ii) *If (X_N) is a net in E with $X_N \leq X_M$ for all $N \leq M$ and $X_N \to X$ for some $X \in E$, then $\sup_{N\in\mathbb{N}(\mathcal{F})} X_N = X$.*

Proof. (i) Since $X = X^+ - X^-$ for all $X \in E$, we see that $E_+ = \{X \in E \mid X^- = 0\}$. In other words, E_+ is the pre–image of $\{0\}$ under the continuous lattice operation $(E, \|\cdot\|) \to (E, \|\cdot\|), X \mapsto X^-$, c.f. remark 2.3. Since $(E, \|\cdot\|)$ is Hausdorff $\{0\}$ is closed, and the assertion follows.

(ii) Throughout, N, M denote elements of $\mathbb{N}(\mathcal{F})$. Let (X_N) in E with $X_N \leq X_M$ for all $N \leq M$ and $X_N \to X$ for some $X \in E$. Since $X_M - X_N \in E_+$ for all $M \geq N$, we see that for all N the net $(X_M - X_N)_{M\geq N}$ in E_+ satisfies $X_M - X_N \to X - X_N$. By (i), E_+ is closed, hence $X - X_N \in E_+$ for all N. Thus, X is an upper bound of the net (X_N). To see that X is the least upper bound of (X_N) take $X' \in E$ with $X' \geq X_N$ for all N. Then, $X' - X_N \in E_+$ for all N and $X' - X_N \to X' - X$ imply $X' - X \in E_+$, whence $X' \geq X$. □

We can now prove theorem 3.1. The proof follows a known pattern; c.f. theorem 9.6 in [1].

Proof. Let $f : E \to L^0$ be a monotone convex local function and $X_0 \in E$. On replacing f by $f(\cdot + X_0) - f(X_0)$ we may assume that $X_0 = 0$ and $f(X_0) = 0$. By way of contradiction, assume that there exists a directed (c.f. lemma 5.1) net (X_N) with $X_N \to 0$ in $(E, \|\cdot\|)$ such that $f(X_N)$ does not converge to 0 in $(L^0, |\cdot|)$. By passing to a subnet of (X_N), we can assume that $f(X_N) \notin W$ for all $N \in \mathbb{N}(\mathcal{F})$ and some neighborhood W of 0 in $(L^0, |\cdot|)$. Consider the neighborhood base of 0 in $(E, \|\cdot\|)$ consisting of all neighborhoods of the form

$$V_N := B_{1/2^N}, \text{ where } N \in \mathbb{N}(\mathcal{F}).$$

Then $V_{N+1} + V_{N+1} \subset V_N$ for all $N \in \mathbb{N}(\mathcal{F})$. Again, by passing to a subnet of (X_N) we can assume that $NX_N \in V_N$ for all $N \in \mathbb{N}(\mathcal{F})$. Next, for each $N \in \mathbb{N}(\mathcal{F})$ define

$$Y_N := \sum_{n\in\mathbb{N}} 1_{\{N=n\}} \sum_{i=1}^{n} i|X_i|,$$

where X_i denotes X_N for $N \equiv i$, $i \in \mathbb{N}$. For all $N, M \in \mathbb{N}(\mathcal{F})$, $A := \{N = n, M = m\}$ and $n, m \in \mathbb{N}$ we observe

$$1_A(Y_{N+M} - Y_N) = 1_A \sum_{i=n+1}^{n+m} i|X_i| \in 1_A(V_{n+1} + V_{n+2} + \cdots + V_{n+m}) \subset 1_A V_n.$$

Hence, $Y_{N+M} - Y_N \in V_N$ for all $N, M \in \mathbb{N}(\mathcal{F})$, that is (Y_N) is Cauchy and so $Y_N \to Y$ for some $Y \in E$. By construction, $0 \leq Y_N \leq Y_M$ for all $N \leq M$, $N, M \in \mathbb{N}(\mathcal{F})$. Hence, (ii) of lemma 5.2 implies $Y = \sup_{N \in \mathbb{N}(\mathcal{F})} Y_N$. Monotonicity of f, convexity of f and $f(0) = 0$ show for all $n \in \mathbb{N}$

$$|f(X_n)| \leq f(|X_n|) \leq \frac{1}{n} f(n|X_n|) \leq \frac{1}{n} f(Y_n) \leq \frac{1}{n} f(Y).$$

Moreover, since f is local we derive for all $N \in \mathbb{N}(\mathcal{F})$

$$|f(X_N)| \leq \frac{1}{N} f(Y).$$

This shows that $f(X_N) \to 0$ in $(L^0, |\cdot|)$ in contradiction to $f(X_N) \notin W$, whence the required continuity. □

5.2 Proof of theorem 3.2

To establish theorem 3.2 we follow the ideas of [26]; however, the proofs are more technical as we work with modules and we have to establish a variety of preliminary results on the way.

Let E be an L^0–module. We recall that a function $p : E \to L^0$ is L^0–positively homogeneous if $p(YX) = Yp(X)$ for all $X \in E$ and $Y \in L^0_+$. p is subadditive if $p(X + X') \leq p(X) + p(X')$ for all $X, X' \in E$. p is L^0–sublinear if it is L^0–positive homogeneous and subadditive.

Lemma 5.3 *Let E be an L^0–normed module, $f : E \to \bar{L}^0$ be a proper L^0–convex function and $X_0 \in \overset{\circ}{\mathrm{dom}} f$. Then, the directional derivative $Df(X_0) : E \to \bar{L}^0$,*

$$Df(X_0)(X) := \underset{Y \in L^0_{++}}{\mathrm{ess.inf}} \frac{f(X_0 + YX) - f(X_0)}{Y}, \tag{5.23}$$

of f at X_0 satisfies:

(i) *For all $X \in E$ and $Y, Y' \in L^0_{++}$ with $Y \leq Y'$*

$$\frac{f(X_0 + YX) - f(X_0)}{Y} \leq \frac{f(X_0 + Y'X) - f(X_0)}{Y'}.$$

In particular, the essential infimum in (5.23) can be taken over all $Y \in L^0_{++}$ with $Y \leq 1$.

(ii) *$Df(X_0)$ is finite valued, that is $Df(X_0)(X) \in L^0$ for all $X \in E$.*

(iii) $Df(X_0)$ *is* L^0*–convex.*

(iv) $Df(X_0)$ *is* L^0*–positively homogeneous.*

(v) $Df(X_0)$ *satisfies the subgradient inequality, that is,* $Df(X_0)(X - X_0) \leq f(X) - f(X_0)$ *for all* $X \in E$.

In particular, $Df(X_0) : E \to L^0$ *is* L^0*–sublinear.*

Proof. Throughout, Y, Y' denote elements of L^0_{++} and $X, X' \in E$.

(i) We have $X_0 + YX = \frac{Y}{Y'}(X_0 + Y'X) + \left(1 - \frac{Y}{Y'}\right) X_0$ so that for L^0–convex f

$$f\left(X_0 + YX\right) \leq \frac{Y}{Y'} f\left(X_0 + Y'X\right) + \left(1 - \frac{Y}{Y'}\right) f(X_0)$$

for all $Y \leq Y'$. Now, divide by Y and rearrange.

(ii) In the same manner as in the proof of lemma 5.41 in [1] p. 187, we derive for L^0–convex f and all $Y \leq 1$

$$|f(X_0 + YZ) - f(X_0)| \leq Y \max\{f(X_0 + Z) - f(X_0), f(X_0 - Z) - f(X_0)\}.$$

whenever $Z \in E$ is such that $X_0 + Z, X_0 - Z \in \mathrm{dom} f$. Since $X_0 \in \overset{\circ}{\mathrm{dom}} f$ there is Y' such that $X_0 + Y'X, X_0 - Y'X \in \mathrm{dom} f$. The assertion now follows from (i).

(iii) By L^0–convexity of f

$$\begin{aligned} & \frac{f(X_0 + Y(ZX + (1 - Z)X')) - f(X_0)}{Y} \\ \leq \quad & Z \frac{f(X_0 + YX) - f(X_0)}{Y} + (1 - Z) \frac{f(X_0 + YX') - f(X_0)}{Y} \end{aligned}$$

for all Y and $Z \in L^0_+$ with $0 \leq Z \leq 1$.

(iv) Let $Z \in L^0_+$. Since f is L^0–convex f is local. With the convention $0/0 := 0$ we derive for local f and for all Y

$$\frac{f(X_0 + YZX) - f(X_0)}{Y} = Z \frac{f(X_0 + YZX) - f(X_0)}{ZY}.$$

(v) For all $Y \leq 1$ we have $X_0 + Y(X - X_0) = YX + (1 - Y)X_0$. Thus, by L^0–convexity

$$\frac{f(X_0 + Y(X - X_0)) - f(X_0)}{Y} \leq f(X) - f(X_0).$$

In view of (i), this yields the assertion. □

The next lemma is a module variant of lemma 1.1 in [23]. A pre stage of lemma 5.4 which, however, only addresses algebraic subdifferentiability of L^0–sublinear functions is established in [20, 21, 22].

Lemma 5.4 *Let E be an L^0–normed module. Any proper L^0–convex function $f : E \to \bar{L}^0$ is algebraically subdifferentiable throughout* $\overset{\circ}{\mathrm{dom}} f$*, that is, for each $X_0 \in \overset{\circ}{\mathrm{dom}} f$ there is an L^0–linear function $\mu : E \to L^0$ such that*

$$\mu(X - X_0) \leq f(X) - f(X_0) \text{ for all } X \in E.$$

Proof. Let $X_0 \in \overset{\circ}{\mathrm{dom}} f$ and denote $p := Df(X_0)$ the directional derivative of f at X_0. By lemma 5.3, $p : E \to L^0$ is L^0–sublinear. We can assume that there is $0 \neq X \in E$. The function $\mu : \mathrm{span}_{L^0}(X) \to L^0$, $\mu(YX) := Yp(X)$ is well defined. Indeed let $Y, Y' \in L^0$ with $YX = Y'X$ and define $A := \{\|X\| = 0\}$. Then, $(Y - Y')X = 0$ and hence

$$0 = \|(Y - Y')X\| = |Y - Y'|\|X\|.$$

Thus, $Y = Y'$ on A^c. Further, $0 = 1_A\|X\| = \|1_A X\|$ and hence $1_A X = 0$. Altogether this yields $Yp(X) = Y'p(X)$ as p is local.

Moreover, $\mu(YX) \leq p(YX)$ for all $Y \in L^0$. Indeed, for $Y \in L^0_+$ this follows from L^0–positive homogeneity of p and the definition of μ. For arbitrary $Y \in L^0$ this follows as p is local and $-p(-X') \leq p(X')$ for all $X' \in E$. Thus, L^0–sublinear p dominates μ on the L^0–submodule generated by X in E and by the Hahn–Banach extension theorem for modules over L^0, c.f. theorem 2.12 in [12] and the references therein, there is an L^0–linear function $\bar{\mu} : E \to L^0$ with $\bar{\mu}(X') \leq p(X')$ for all $X' \in E$. By (v) of lemma 5.3, $\bar{\mu}$ is as required. ☐

Lemma 5.5 below is a module variant of theorem 3.3 in [25]. Its proof draws from some results in [12]. Therefore, we adopt the respective notation and denote by (X_α) arbitrary nets in L^0. Further, we define the essential limit inferior $\mathrm{ess.liminf}_\alpha X_\alpha$ of a net (X_α) in L^0 by

$$\underset{\alpha}{\mathrm{ess.liminf}}\, X_\alpha := \underset{\alpha}{\mathrm{ess.sup}}\, \underset{\beta \geq \alpha}{\mathrm{ess.inf}}\, X_\beta.$$

Lemma 5.5 *Let E be an L^0–normed module. If a proper L^0–convex function $f : E \to \bar{L}^0$ satisfies*

$$f(X) \leq \underset{\alpha}{\mathrm{ess.liminf}}\, f(X_\alpha) \tag{5.24}$$

for all nets (X_α) with $X_\alpha \to X$ for some $X \in \overset{\circ}{\mathrm{dom}} f$, then f is continuous throughout $\overset{\circ}{\mathrm{dom}} f$.

Proof. Let $X_0 \in \overset{\circ}{\mathrm{dom}} f$. On replacing f by $f(\cdot + X_0)$ we can assume that $X_0 = 0$. We will show that f is continuous at 0.

Since $0 \in \overset{\circ}{\mathrm{dom}} f$ let $N \in \mathbb{N}(\mathcal{F})$ such that $B_{1/N} \subset \overset{\circ}{\mathrm{dom}} f$. We define $\widetilde{f} : E \to \bar{L}^0$,

$$\widetilde{f}(X) := f(X) + \infty 1_{\{\|X\| > 1/N\}},$$

with the convention $0 \cdot \infty = 0$. Note that $1_{\{\|X\| \leq 1/N\}} X \in B_{1/N}$. Then, $\widetilde{f}$ coincides with f on the neighborhood $B_{1/N}$ of $0 \in E$. Therefore, it suffices to show that $\widetilde{f}$ is continuous at 0. Indeed, if (X_α) is a net in E with $X_\alpha \to 0$, then eventually $X_\alpha \in B_{1/N}$ for large α. Continuity of $\widetilde{f}$ then shows $f(X_\alpha) = \widetilde{f}(X_\alpha) \to \widetilde{f}(0) = f(0)$.

To establish continuity of $\widetilde{f}$ we show that $\widetilde{f}$ is proper L^0–convex (hence local) and satisfies

$$\widetilde{f}(X) \leq \operatorname*{ess.liminf}_{\alpha} \widetilde{f}(X_\alpha) \tag{5.25}$$

for all nets (X_α) with $X_\alpha \to X$ for some $X \in E$. We then apply some results of [12]. Properness follows from that of f. For L^0–convexity, let $X, X' \in E$, $Y \in L^0_+$, $0 \leq Y \leq 1$ and observe on $A := \{\|X\| > 1/N\} \cup \{\|X'\| > 1/N\}$

$$\widetilde{f}(YX + (1-Y)X') \leq Y\widetilde{f}(X) + (1-Y)\widetilde{f}(X'),$$

as the righthand side is ∞ on A. Since $A^c = \{\|X\| \leq 1/N\} \cap \{\|X'\| \leq 1/N\} \subset \{\|YX + (1-Y)X'\| \leq 1/N\}$ the required inequality on A^c follows from L^0–convexity of f. To establish (5.25) let $(X_\alpha) \subset E$ with $X_\alpha \to X$, $X \in E$. Define $\widetilde{X}_\alpha := 1_{\{\|X\| \leq 1/N\}} X_\alpha$. Then $\widetilde{X}_\alpha \to 1_{\{\|X\| \leq 1/N\}} X \in B_{1/N} \subset \overset{\circ}{\mathrm{dom}} f$. Hence,

$$\begin{aligned} \operatorname*{ess.liminf}_{\alpha} \widetilde{f}(\widetilde{X}_\alpha) &\geq \operatorname*{ess.liminf}_{\alpha} f(\widetilde{X}_\alpha) && (5.26)\\ &\geq f(1_{\{\|X\| \leq 1/N\}} X), && (5.27) \end{aligned}$$

where (5.26) is by definition of $\widetilde{f}$ and (5.27) is by (5.24). Multiplying by $1_{\{\|X\| \leq B_{1/N}\}}$ yields (for local $\widetilde{f}$)

$$1_{\{\|X\| \leq 1/N\}} \operatorname*{ess.liminf}_{\alpha} \widetilde{f}(\widetilde{X}_\alpha) \geq 1_{\{\|X\| \leq 1/N\}} \widetilde{f}(X).$$

Since $X_\alpha \to X$ we have $\|X_\alpha\| > 1/N$ on $\{\|X\| > 1/N\}$ for sufficiently large α. Thus, $\widetilde{f}(X_\alpha) = \infty = \widetilde{f}(X)$ on $\{\|X\| > 1/N\}$ for sufficiently large α, that is, we established (5.25).

Altogether, $\widetilde{f} : E \to \bar{L}^0$ is proper L^0–convex and satisfies (5.25). By proposition 3.4 in [12] in conjunction with lemma 3.9 in [12], $\widetilde{f}$ is continuous throughout $\overset{\circ}{\mathrm{dom}} \widetilde{f}$. In particular, $\widetilde{f}$ is continuous at 0. Thus, f is so. □

We can now prove theorem 3.2.

Proof. Let $X_0 \in \overset{\circ}{\mathrm{dom}} f$. By lemma 5.4, there is L^0–linear $\mu : E \to L^0$ such that

$$\mu(X - X_0) \leq f(X) - f(X_0) \text{ for all } X \in E. \tag{5.28}$$

μ is monotone as f is so. Indeed, assume, by way of contradiction, there is $X \in E_+$ such that $P[\mu X < 0] > 0$. Then, by (5.28), $f(X_0 - X) \geq f(X_0) - \mu X$. Hence, $P[f(X_0 - X) > f(X_0)] > 0$ in contradiction to monotonicity of f. Consequently, μ is monotone. In particular, μ is L^0–convex and so it is continuous by theorem 3.1. Thus, μ is a subgradient of f at X_0.

To establish continuity of f at X_0 observe that (5.28) together with the continuity of μ implies that f satisfies

$$\operatorname*{ess.limsup}_{\alpha} f(X) - f(X_\alpha) \leq \operatorname*{ess.limsup}_{\alpha} \mu(X - X_\alpha) = 0,$$

and whence (5.24). Thus, f is continuous throughout $\overset{\circ}{\mathrm{dom}} f$ due to lemma 5.5. □

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