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Abstract

This paper considers the pricing and hedging of collateralized debt obligations (CDOs). CDOs are complex derivatives on a pool of credits which we choose to analyse in the top down model proposed in Filipović et al. [4]. We reflect on the implied forward rates and bring them in connection with the top-down framework in Lipton and Shelton [10] and Schönbucher [13]. Moreover, we derive variance-minimizing hedging strategies for hedging single tranches with the full index. The hedging strategies are given for the general case. We compute them also explicitly for a parsimonious one-factor affine model.

1 Introduction

In this paper we gradually develop a general formula for the variance-minimizing hedging strategy for a single tranche CDO within the top-down model framework recently developed in Filipović et al. [4].

In the last decade the markets of collateralized debt obligations (CDOs) have witnessed a tremendeous activity and many models have been developed for pricing and some for hedging. The current market turmoil however illustrates that mostly used approaches - typically static models, such as the Gaussian copula model - are not able to capture the dynamic nature of the model. This is important for consistent pricing and even more important for hedging. In this paper we concentrate on the dynamic top down model proposed in Filipović et al. [4] and derive variance-minimizing hedging strategies. For an overview of credit risk modelling we refer to the respective chapters in [11].

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The most liquidly traded CDOs are those based on so-called indices. In 2004 the CDX in North America and the iTraxx in Europe have been created. For example, the iTraxx Europe consists of the 125 most liquid investement grade coporate credit default swaps. Besides the index *single tranche CDOs* (STCDOs) are liquidly traded. The STCDOs allow to invest in parts of the CDO, so-called tranches, see Section 4 below for details. In this paper we analyze the hedging of a STCDO with the index and derive the variance-minimizing hedging strategy. Besides the hedging strategy for the general form, we compute the hedging strategy explicitly in a simple one-factor affine model. This simple model is dynamic and allows to fit any given initial term structure of CDOs perfectly. For related articles on the dynamic hedging of credit portfolio products and CDOs we refer the reader to [1].

We assume a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ satisfying the usual conditions, and where $\mathbb{Q}$ denotes a risk-neutral pricing measure. We consider a portfolio of credits with an overall outstanding notional normalized to 1. We follow a top-down approach and assume that the loss process

$$L_t = \sum_{s \leq t} \Delta L_s = \int_0^t \int_{(0,1]} x \, \mu(ds, dx)$$

be an $\mathcal{I} := [0,1]$-valued non-decreasing marked point process with absolutely continuous compensator $\nu(t, dx)dt$. We let $\mu(dt, dx)$ be the integer-valued random measure associated with the jumps of L. The case of finitely many loss fractions $\mathcal{I} = \{\frac{i}{n} : i = 0, \ldots, n\}$ is a special case of the setup.

We then define the (T, x)-bond which pays $1_{\{L_T \leq x\}}$ at maturity T, for $x \in [0,1]$. In other words, this is a zero-recovery defaultable bond. Its arbitrage-free price at time $t \leq T$ is denoted by $P(t, T, x)$. By construction, $P(t,T) = P(t, T, 1)$ is the risk-free zero coupon bond. As a consequence, $P(t, T, \cdot)/P(t, T) = \mathbb{E}_{\mathbb{Q}^T}[1_{\{L_T \leq x\}} \mid \mathcal{F}_t]$ is just the $\mathcal{F}_t$-conditional cumulative distribution function of L_T under the T-forward measure $\mathbb{Q}^T$. Moreover, any European type T-claim on the loss process with absolutely continuous payoff function $H(L_T)$ can be decomposed into an infinite linear combination of (T, x)-bond payoffs:

$$H(L_T) = H(1) - \int_{(0,1]} H'(x) 1_{\{L_T \leq x\}} \, dx.$$

Here with H' we denote the Dini derivative of H, which is known to be locally integrable in x (see e.g. [14, Theorem 7.29]). As a simple consequence the price π_t of the claim at any time $t \leq T$ is given by

$$\pi_t = H(1)P(t,T) - \int_{(0,1]} H'(x) P(t, T, x) \, dx. \tag{1}$$

This paper is structured as follows. In Section 2 we derive the arbitrage free (T, x)-bond dynamics, and obtain a no-arbitrage criterion as proposed in Filipović et al. [4]. In Section 3 we reflect on the implied forward rates and bring them in connection with the top-down framework in Lipton and Shelton

[10] and Schönbucher [13]. In Section 4 we calculate the gains processes from a single tranche CDO with arbitrary detachment points. This includes the entire index in particular. It is understood that the index can be replicated by holding the respective positions in the constituent CDS. In Section 5 we then derive the variance-minimizing hedging strategy for any STCDO in terms of the index in general. This is then explicitly computed for a simple one-factor affine model in Section 6.

2 (T,x)-Bond Dynamics

In this section we recap the framework for arbitrage-free term structure movements as laid out in [4]. The (T,x)-bond price is decomposed into default event and market risk:

$$P(t,T,x) = 1_{\{L_t \le x\}} \mathrm{e}^{-\int_t^T f(t,u,x)du}.$$

We assume that, for all (T,x), the (T,x)-forward rate process $f(t,T,x)$, $t \le T$, follows a semimartingale of the form

$$\begin{aligned} f(t,T,x) = f(0,T,x) + \int_0^t a(s,T,x)ds + \int_0^t b(s,T,x)^\top dW_s \\ + \int_0^t \int_{(0,1]} c(s,T,x;y)\,\mu(ds,dy) \end{aligned} \tag{2}$$

where W is some d-dimensional Brownian motion. To justify the subsequent stochastic analysis we make the following technical assumptions, where $\mathcal{O}$ and $\mathcal{P}$ denote the optional and predictable σ-algebra on $\Omega \times \mathbb{R}_+$, respectively:

(A1) The initial forward curve $f(0,T,x)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I})$-measurable, and locally integrable:

$$\int_0^T |f(0,u,x)|\,du < \infty \quad \text{for all } (T,x).$$

The drift parameter $a(t,T,x)$ is $\mathbb{R}$-valued $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I})$-measurable, and locally integrable:

$$\int_0^T \int_0^T |a(t,u,x)|\,dt\,du < \infty \quad \text{for all } (T,x).$$

The volatility parameter $b(t,T,x)$ is $\mathbb{R}^d$-valued $\mathcal{O}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathcal{I})$-measurable, and locally bounded in the sense that

$$\sup_{0\le t\le u\le T} \|b(t,u,x)\| < \infty \quad \text{for all } (T,x).$$

The contagion parameter $c(t,T,x;y)$ is $\mathbb{R}$-valued $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I}) \otimes \mathcal{B}((0,1])$-measurable, and locally bounded in the sense that

$$\sup_{0\le t\le u\le T,\, y\in(0,1]} |c(t,u,x;y)| < \infty \quad \text{for all } (T,x).$$

Under these conditions, the integrals in (2) are well defined in particular. Moreover, they imply that the risk-free short rate $r_t = f(t,t,1)$ has a progressive version and satisfies $\int_0^T |r_t|\,dt < \infty$ for all T, see e.g. [5, Corollary 6.3]. Hence the risk-free numeraire $\mathrm{e}^{\int_0^t r_s ds}$ is well defined. However, these conditions do not imply that spreads $f(t,T,x_1) - f(t,T,x_2)$, for $x_1 < x_2$ are nonnegative in general. Nonnegativity of spreads has to be established from case to case, such as in the affine model in Section 6 below.

Denote the discounted (T,x)-bond price by

$$Z(t,T,x) = \mathrm{e}^{-\int_0^t r_s\,ds} P(t,T,x).$$

Lemma 2.1. *Under (A1) the implied dynamics of the discounted (T,x)-bond price process is given by*

$$\frac{dZ(t,T,x)}{Z(t-,T,x)} = \alpha(t,T,x)\,dt + \beta(t,T,x)\,dW_t + \int_{(0,1]} \gamma(t,T,x,\xi)(\mu(dt,d\xi) - \nu(t,d\xi)dt)$$

where

$$\begin{aligned}
\alpha(t,T,x) &= -r_t - \lambda(t,x) + f(t,t,x) - \int_t^T a(t,u,x)\,du \\
&\quad + \frac{1}{2}\left\| \int_t^T b(t,u,x)\,du \right\|^2 \\
&\quad + \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\,du} - 1 \right) 1_{\{L_t + y \le x\}}\,\nu(t,dy) && (3) \\
\beta(t,T,x) &= -\int_t^T b(t,u,x)^\top du && (4) \\
\gamma(t,T,x,\xi) &= \mathrm{e}^{-\int_t^T c(t,u,x;\xi)\,du} 1_{\{L_{t-} + \xi \le x\}} - 1, && (5)
\end{aligned}$$

and we define

$$\lambda(t,x) = \int_{(0,1]} 1_{\{L_t + y > x\}}\,\nu(t,dy). \quad (6)$$

The corresponding stochastic exponential representation of $Z(t,T,x)$ reads

$$\begin{aligned}
Z(t,T,x) &= Z(0,T,x) \exp\left(\int_0^t \alpha(s,T,x)\,ds \right) \\
&\quad \times \exp\left(\int_0^t \beta(s,T,x)\,dW_s - \frac{1}{2}\int_0^t \|\beta(s,T,x)\|^2\,ds \right) \\
&\quad \times \exp\left(-\int_0^t \gamma(s,T,x,\xi)\,\nu(s,d\xi)ds \right) \prod_{s \le t} \left(1 + \gamma(s,T,x,\Delta L_s) 1_{\{\Delta L_s > 0\}} \right). \quad (7)
\end{aligned}$$

In particular, we have $\Delta Z(t,T,x)/Z(t-,T,x) = \gamma(t,T,x,\Delta L_t)1_{\{\Delta L_t > 0\}}$, which equals -1 if the loss process crosses level x at t, that is, $L_{t-} \le x < L_t$. This is consistent with the fact that $Z(t,T,x) = 1_{\{L_t \le x\}} Z(t,T,x)$ for all t.

Remark 2.2. *Note that* $\lambda(t,x)$ *in* (6) *is nothing but the intensity of the* x*-crossing time* $\tau_x = \inf\{t \mid L_t > x\}$ *of* L*. Indeed, this becomes obvious since we can write*

$$1_{\{\tau_x \le t\}} = 1_{\{L_t > x\}} = \int_0^t \int_{(0,1]} 1_{\{L_{s-}+y>x\}} 1_{\{L_{s-} \le x\}} \, \mu(ds, dy). \tag{8}$$

Moreover, conversely to (6)*,* $\lambda(t,x)$ *uniquely determines* $\nu(t,dx)$ *via*

$$\nu(t,(0,x]) = \lambda(t, L_t) - \lambda(t, L_t + x), \quad x \in (0,1], \tag{9}$$

where we define $\lambda(t,x) = 0$ *for* $x \ge 1$*. Furthermore,* $\lambda(t,x)$ *is decreasing in* x *for any* t *by* (9)*.*

As a corollary of Lemma 2.1, we obtain the no-arbitrage drift condition from [4, Theorem 3.2]:

Corollary 2.3. *No-arbitrage, that is,* $Z(t,T,x)$ *is a local martingale for all* (T,x)*, holds if and only if*

$$\begin{aligned} \int_t^T a(t,u,x)\, du &= \frac{1}{2} \left\| \int_t^T b(t,u,x)\, du \right\|^2 \\ &\quad + \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} - 1 \right) 1_{\{L_t + y \le x\}} \, \nu(t, dy), \end{aligned} \tag{10}$$

$$r_t + \lambda(t,x) = f(t,t,x) \tag{11}$$

on $\{L_t \le x\}$*,* $dt \otimes d\mathbb{Q}$*-a.s. for all* (T,x)*.*

Proof of Lemma 2.1. As in the proof of [4, Theorem 3.2] we decompose $p(t,T,x) = \mathrm{e}^{-\int_t^T f(t,u,x)\, du}$ as

$$\begin{aligned} \frac{dp(t,T,x)}{p(t-,T,x)} = &\left\{ f(t,t,x) - \int_t^T a(t,u,x)\, du + \frac{1}{2} \left\| \int_t^T b(t,u,x)\, du \right\|^2 \right. \\ &\left. + \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} - 1 \right) \nu(t,dy) \right\} dt \\ &- \int_t^T b(t,u,x)^\top \, du \cdot dW_t \\ &+ \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} - 1 \right) \left(\mu(dt,dy) - \nu(t,dy)\, dt \right). \end{aligned}$$

Note that we can write, as in (8)

$$1_{\{L_t \le x\}} = 1 + \int_0^t \int_{(0,1]} (-1_{\{L_{s-}+y>x\}} 1_{\{L_{s-} \le x\}}) \mu(ds, dy).$$

Integration by parts thus gives

$$
\begin{aligned}
&d(1_{\{L_t \le x\}} p(t,T,x)) \\
&= 1_{\{L_{t-} \le x\}}\, dp(t,T,x) + p(t-,T,x)\, d1_{\{L_t \le x\}} + d[1_{\{L_t \le x\}}, p(t,T,x)] \\
&= 1_{\{L_{t-} \le x\}}\, p(t-,T,x)\, \frac{dp(t,T,x)}{p(t-,T,x)} \\
&\quad + p(t-,T,x) \int_{(0,1]} (-1_{\{L_{t-}+y>x\}} 1_{\{L_{t-} \le x\}})\, \mu(dt,dy) \\
&\quad + p(t-,T,x) \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} - 1 \right) (-1_{\{L_{t-}+y>x\}} 1_{\{L_{t-} \le x\}})\, \mu(dt,dy) \\
&= 1_{\{L_{t-} \le x\}}\, p(t-,T,x) \left(\frac{dp(t,T,x)}{p(t-,T,x)} - \int_{(0,1]} \mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} 1_{\{L_{t-}+y>x\}}\, \mu(dt,dy) \right) \\
&= 1_{\{L_{t-} \le x\}}\, p(t-,T,x)(D(t)\, dt + dN(t))
\end{aligned}
$$

where the local martingale part is given by

$$
\begin{aligned}
dN(t) = &- \int_t^T b(t,u,x)^\top\, du \cdot dW_t \\
&+ \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} 1_{\{L_{t-}+y \le x\}} - 1 \right) (\mu(dt,dy) - \nu(t,dy)\, dt),
\end{aligned}
$$

and the drift part is

$$
\begin{aligned}
D(t) &= f(t,t,x) - \int_t^T a(t,u,x)\, du + \frac{1}{2} \left\| \int_t^T b(t,u,x)\, du \right\|^2 \\
&\quad + \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} - 1 \right) \nu(t,dy) - \int_{(0,1]} \mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} 1_{\{L_t+y>x\}}\, \nu(t,dy) \\
&= f(t,t,x) - \int_t^T a(t,u,x)\, du + \frac{1}{2} \left\| \int_t^T b(t,u,x)\, du \right\|^2 \\
&\quad + \int_{(0,1]} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\, du} - 1 \right) 1_{\{L_t+y \le x\}}\, \nu(t,dy) - \int_{(0,1]} 1_{\{L_t+y>x\}}\, \nu(t,dy).
\end{aligned}
$$

Note that the last summand equals $\lambda(t,x)$. Discounting by $\mathrm{e}^{\int_0^t r_s\, ds}$ yields (3)–(5). The stochastic exponential representation (7) is standard, see e.g. [9, Section I.4f]. □

3 (T,x)-Forward Rates

In this intermediary section, we briefly reflect on the corresponding forward rates and discuss their relation to some other top-down approaches. Note that $f(t,T) = f(t,T,1)$ is the risk-free forward rate.

Equation (11) states that the no-arbitrage property of $P(t,T,x)$ implies that the short rates equal the sum of risk-free short rate plus τ_x-intensity. In a heuristic manner, we can carry this property over to the forward rates:

$$f(t,T,x) - f(t,T) = \lim_{\Delta T \to 0} \frac{1}{\Delta T} \mathbb{Q}^T[L_{T+\Delta T} > x \mid L_T \le x,\, \mathcal{F}_t] \tag{12}$$

where $\mathbb{Q}^T$ denotes the T-forward measure. Whence $f(t,T,x) - f(t,T)$ is the $\mathbb{Q}^T$-forward transition rate prevailing at date t for L to jump at T from below or equal to above level x.

Indeed, for the sake of simplicity, let us for the rest of this section assume zero risk-free rates $f(t,T) = r_t = 0$. Then, assuming a continuous term structure $T \mapsto f(t,T,x)$, we obtain

$$\begin{aligned} f(t,T,x) &= \lim_{\Delta T \to 0} \frac{1}{\Delta T} \frac{P(t,T,x) - P(t,T+\Delta T,x)}{P(t,T,x)} \\ &= \lim_{\Delta T \to 0} \frac{1}{\Delta T} \frac{\mathbb{E}[1_{\{L_T \le x\}} - 1_{\{L_{T+\Delta T} \le x\}} \mid \mathcal{F}_t]}{\mathbb{E}[1_{\{L_T \le x\}} \mid \mathcal{F}_t]} \\ &= \lim_{\Delta T \to 0} \frac{1}{\Delta T} \frac{\mathbb{E}[1_{\{L_T \le x\}} 1_{\{L_{T+\Delta T} > x\}} \mid \mathcal{F}_t]}{\mathbb{E}[1_{\{L_T \le x\}} \mid \mathcal{F}_t]} \\ &= \lim_{\Delta T \to 0} \frac{1}{\Delta T} \mathbb{Q}[L_{T+\Delta T} > x \mid L_T \le x,\, \mathcal{F}_t], \end{aligned}$$

which is (12).

3.1 Relation to the Top-Down Model in Lipton–Shelton [10]

Our approach can be brought in connection with the top-down model in Lipton and Shelton [10, Section 6.2], which is based on Schönbucher [13], as follows.

Suppose, as in [10], that the risk-free rates $f(t,T) = r_t = 0$, and that the credit portfolio consists of N_c credits with nominal 1 each, and the loss process can only assume fractions i/N_c, $i = 0, \dots, N_c$. Then $f(t,T,x) = f(t,T,i/N_c)$ for all $i/N_c \le x < (i+1)/N_c$. If, moreover, there are no simultaneous defaults, with the sole exception of a systemic default; that is, L_T can either jump from i/N_c to $(i+1)/N_c$ or to 1. Then, view of (12), we obtain with the notation of Lipton and Shelton [10, Section 6.2]:

$$\begin{aligned} f(t,T,x) &= \frac{\sum_{l \le [N_c x]} p_{l|m}(t,T) \left(\sum_{n > [N_c x]} a_{ln}(t,T) \right)}{\sum_{l \le [N_c x]} p_{l|m}(t,T)} \\ &= 1_{\{[N_c x] < N_c\}} \frac{p_{[N_c x]|m}(t,T) a_{[N_c x]}(t,T) + \sum_{l \le [N_c x]} p_{l|m}(t,T) b(t,T)}{\sum_{l \le [N_c x]} p_{l|m}(t,T)} \end{aligned} \tag{13}$$

given that $L_t = m/N_c$, where $[z] = \max\{i \in \mathbb{Z} \mid i \le z\}$ denotes the largest integer smaller or equal to z. Here $a_{ln}(t,T)$, $a_l(t,T) = a_{l,l+1}(t,T)$ and $b(t,T)$ denote the forward transition rates prevailing at date t for L to jump at T

from l/N_c to n/N_c, $(l+1)/N_c$ and 1 (systemic default), respectively. Moreover, $p_{l|m}(t,T) = \mathbb{Q}[L_T = l/N_c \mid L_t = m/N_c,\, \mathcal{F}_t]$.

For the short rates, we thus obtain

$$f(t,t,x) = 1_{\{x<1\}} \left(a_{[N_c x]}(t,t) 1_{\{L_t = [N_c x]/N_c\}} + b(t,t) \right) \tag{14}$$

on $\{L_t \leq x\}$. Combining this with (9) and (11), we obtain the following obvious relation:

$$\begin{aligned} \nu(t,(0,x]) &= f(t,t,L_t) - f(t,t,L_t + x) \\ &= 1_{\{L_t<1\}} \left(a_{[N_c L_t]}(t,t) 1_{\{x \geq 1/N_c\}} + b(t,t) 1_{\{L_t + x \geq 1\}} \right). \end{aligned}$$

4 Single Tranche CDOs (STCDOs)

STCDO are the most liquidly traded CDO derivatives. For the iTraxx Europe, the tranches 0-3%, 3-6%, 6-9%, 9-12%, and 12-22% are traded. Also special sub-indices exist, e.g. the iTraXX Europe HiVol which contains the 30 highes spread entities from the iTraxx Europe. We first describe the payment structure of STCDOs and then compute the dynamics of the related gains processes. A STCDO issued at $t = 0$ is specified by

- a number of coupon payment dates $0 < T_1 < \cdots < T_n$ (T_n is the maturity of the STCDO)
- a *tranche* with lower and upper detachment points $x_1 < x_2$ in $\mathcal{I}$,
- a fixed swap rate s_0.

We write

$$H(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{(x_1,x_2]} 1_{\{x \leq y\}} dy.$$

An investor in this STCDO

- receives $s_0 H(L_{T_i})$ at T_i, $i = 1, \ldots, n$ (payment leg),
- pays $-dH(L_t) = H(L_{t-}) - H(L_t)$ at any time $t \leq T_n$ where $\Delta L_t \neq 0$ (default leg).

The accumulated discounted cash flow by time t thus equals

$$\begin{aligned}
A_t &= s_0 \sum_{T_i \le t} \mathrm{e}^{-\int_0^{T_i} r_s\, ds} H(L_{T_i}) + \int_0^t \mathrm{e}^{-\int_0^u r_s\, ds}\, dH(L_u) \\
&= s_0 \sum_{T_i \le t} \mathrm{e}^{-\int_0^{T_i} r_s\, ds} H(L_{T_i}) \\
&\quad + \mathrm{e}^{-\int_0^t r_s\, ds} H(L_t) - H(L_0) + \int_0^t r_u \mathrm{e}^{-\int_0^u r_s\, ds} H(L_u)\, du \\
&= \int_{(x_1,x_2]} \Big\{ s_0 \sum_{T_i \le t} \mathrm{e}^{-\int_0^{T_i} r_s\, ds} 1_{\{L_{T_i} \le x\}} \\
&\quad + \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}} - 1_{\{L_0 \le x\}} + \int_0^t r_u \mathrm{e}^{-\int_0^u r_s\, ds} 1_{\{L_u \le x\}}\, du \Big\} dx
\end{aligned} \tag{15}$$

where we have integrated by parts the default leg cash flow.

Throughout we shall assume that the accumulated discounted cash flow is square integrable:

(A2) $A_t \in L^2$ for all $t \le T_n$.

The discounted time t spot value of the STCDO is given by the expectation of future discounted cash-flows and computes to

$$\begin{aligned}
\Gamma_t &= \mathbb{E}\left[A_{T_n} - A_t \mid \mathcal{F}_t\right] \\
&= \int_{(x_1,x_2]} \left\{ s_0 \sum_{t < T_i} Z(t, T_i, x) - \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}} + Z(t, T_n, x) + \delta(t, x) \right\} dx
\end{aligned}$$

where

$$\delta(t, x) = \int_t^{T_n} \mathbb{E}\left[r_u \mathrm{e}^{-\int_0^u r_s\, ds} 1_{\{L_u \le x\}} \mid \mathcal{F}_t \right] du.$$

The par swap rate at time t, which is quoted in the market, is defined as the swap rate which gives the STCDO zero value if entered at t, that is, $\Gamma_t = 0$. It computes to

$$s_t = \frac{\int_{(x_1,x_2]} \left\{ \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}} - Z(t, T_n, x) - \delta(t, x) \right\} dx}{\int_{(x_1,x_2]} \sum_{t < T_i} Z(t, T_i, x)\, dx}.$$

The discounted t spot value Γ_t can thus be expressed as

$$\Gamma_t = (s_0 - s_t) \int_{(x_1,x_2]} \sum_{t < T_i} Z(t, T_i, x)\, dx.$$

The gains process from holding the STCDO equals

$$G_t = A_t + \Gamma_t = \mathbb{E}\left[A_{T_n} \mid \mathcal{F}_t\right]$$
$$= \int_{(x_1,x_2]} \Bigg\{ s_0 \left(\sum_{T_i \le t} \mathrm{e}^{-\int_0^{T_i} r_s\, ds} 1_{\{L_{T_i} \le x\}} + \sum_{t<T_i} Z(t, T_i, x) \right)$$
$$- 1_{\{L_0 \le x\}} + Z(t, T_n, x) + \int_0^t r_u \mathrm{e}^{-\int_0^u r_s\, ds} 1_{\{L_u \le x\}}\, du + \delta(t,x) \Bigg\} dx.$$

In view of **(A2)**, G is a square integrable martingale, and it satisfies

$$dG_t = \int_{(x_1,x_2]} \Bigg\{ s_0 \sum_{t<T_i} dZ(t, T_i, x)$$
$$+ dZ(t, T_n, x) + r_t \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}}\, dt + d\delta(t,x) \Bigg\} dx. \quad (16)$$

The above analysis, including assumption **(A2)**, prevails for any tranche $(x_1, x_2]$. We shall write accordingly

$$s_t = s_t^{(x_1,x_2]}, \quad A_t = A_t^{(x_1,x_2]}, \quad \Gamma_t = \Gamma_t^{(x_1,x_2]}, \quad G_t = G_t^{(x_1,x_2]}$$

and formulate the modified version of (A2).

(A2') Assume that for each $x_1 < x_2$ with $x_1, x_2 \in \mathcal{I}$ it holds that $A_t^{(x_1,x_2]} \in L^2$ for all $t \le T_n$.

Here is a sufficient condition:

Lemma 4.1. *If* $\sup_{t \le T_n} \mathrm{e}^{-\int_0^t r_s\, ds} \in L^2$ *then* **(A2')** *is satisfied. This holds in particular if* r *is nonnegative.*

Proof. Follows directly from the representation of A_t in Equation (15). □

Note that $x_1 = 0$ and $x_2 = 1$ corresponds to the entire index, which is composed by some balanced portfolio of single name CDS. It is understood that the index can be replicated by holding the respective positions in the constituent CDS. In the following section, we provide a risk minimizing hedging strategy for any STCDO in terms of the index.

5 Variance-minimizing Hedging

As proposed by Cont and Kan [1], we now derive the variance-minimizing hedging strategy of the $(x_1, x_2]$-tranche STCDO with the index. Recall that, by assumption **(A2')**, the gains process $G_t^{(x_1,x_2]}$ for any tranche $(x_1, x_2]$ is a square integrable martingale.

For a pair of square integrable martingales M and N we denote by $\langle M, N\rangle$ their predictable quadratic covariation (see [9, Section I.4a]). Notice that

$$\mathbb{E}[M_{T_n} N_{T_n}] = \mathbb{E}[\langle M, N\rangle_{T_n}] + \mathbb{E}[M_0 N_0]$$

defines a scalar product on the space of square integrable martingales on $[0, T_n]$. The following can thus be seen as an orthogonal projection statement.

Theorem 5.1. *Assume that* **(A2')** *holds. For any time interval* $0 \le t < T \le T_n$*, the self-financing strategy*

$$\phi^* = \frac{d\langle G^{(x_1,x_2]}, G^{(0,1]}\rangle}{d\langle G^{(0,1]}\rangle}$$

along with the initial capital $c^* = G_t^{(x_1,x_2]}$ *is the unique minimizer of the quadratic hedging error*

$$\text{ess inf}_{c,\phi}\, \mathbb{E}\left[\left(c + \int_t^T \phi_s\, dG_s^{(0,1]} - G_T^{(x_1,x_2]}\right)^2 \mid \mathcal{F}_t\right].$$

Here the essential infimum is taken over all $c \in L^2(\mathcal{F}_t)$ *and predictable processes* ϕ *with*

$$\mathbb{E}\left[\int_0^{T_n} \phi_s^2\, d\langle G^{(0,1]}\rangle_s\right] < \infty. \tag{17}$$

This strategy is referred to as the variance-minimizing strategy.

Proof. By Assumption **(A2')** the process given by $G_t^{(x_1,x_2]} = \mathbb{E}(A_{T_n}^{(x_1,x_2]}|\mathcal{F}_t)$ is a square-integrable martingale. Hence, by Proposition 10.4 in [2] it can be decomposed in the so-called Goultchuk-Kunita-Watanabe decomposition:

$$G_t^{(x_1,x_2]} = G_0^{(x_1,x_2]} + \int_0^t \xi_s dG_s^{(0,1]} + G'_t. \tag{18}$$

Here G' is a square integrable martingale with mean zero and orthogonal to $G^{(0,1]}$ in the sense that $\langle G', G^{(0,1]}\rangle = 0$. Theorem 2.1 in Møller [12] states that the variance-minimizing strategy is given by the term ξ in Equation (18). Orthogonality of G' and $G^{(0,1]}$ yields

$$d\langle G^{(x_1,x_2]}, G^{(0,1]}\rangle_t = \xi_t d\langle G^{(0,1]}\rangle_t$$

and the represenation of $\xi = \phi^*$ follows. The initial cost, given by Equation (2.3) in Møller [12], equals $A_t^{(x_1,x_2]} + \Gamma_t^{(x_1,x_2]} = G_t^{(x_1,x_2]}$. □

Remark 5.2. *The variance-minimizing hedging strategy minimizes a quadratic risk directly under the risk-neutral measure* Q*. This approach has also pursued in a number of different papers ([1], [6], [7]) It is particularly useful when the distribution under the objective measure is difficult to obtain. As markets of CDOs are quite young, only few historical data is available which makes statistical estimation difficult.*

Note that the variance-minimizing strategy ϕ^* does not depend on the reference time interval $[t, T]$, while it does on the tranche $(x_1, x_2]$ of course. Intuitively speaking, ϕ_t^* minimizes, locally for all t,

$$\mathbb{E}\left[\left(dG_t^{(x_1,x_2]} - \phi_t \, dG_t^{(0,1]}\right)^2 \mid \mathcal{F}_t\right]$$

among all predictable ϕ_t which satisfy (17).

In the following, we compute ϕ^* for model specifications in various degrees of generality.

5.1 Deterministic Risk Free Rates

In this section we assume deterministic interest rates and derive the respective variance-minimizing hedging strategy in detail. We first compute the necessary terms of Equation (16). The gains process and the hedging strategy follow.

Lemma 5.3. *If the risk free interest rates r_t are deterministic, then*

$$d\delta(t,x) = \mathcal{B}(t,x)\,dW_t + \int_{(0,1]} \mathcal{C}(t,x,\xi)\,(\mu(dt,d\xi) - \nu(t,d\xi)dt) - r_t \mathrm{e}^{-\int_0^t r_s} 1_{\{L_t \leq x\}}$$

where

$$\mathcal{B}(t,x) = \int_t^{T_n} r_u Z(t,u,x)\beta(t,u,x)\,du$$
$$\mathcal{C}(t,x,\xi) = \int_t^{T_n} r_u Z(t-,u,x)\gamma(t,u,x,\xi)\,du.$$

Proof. If the risk free interest rates r_t are deterministic, we obtain

$$\delta(t,x) = \int_t^{T_n} r_u Z(t,u,x)\,du.$$

Using a stochastic Fubini argument as in the proof of [4, Theorem 3.2], we transform in the following

$$\int_t^{T_n} \int_0^t \cdots ds\,du = \int_0^t \int_t^{T_n} \cdots du\,ds = \int_0^t \int_s^{T_n} \cdots du\,ds - \int_0^t \int_0^u \cdots ds\,du,$$

and similarly for $dW_s\,du$ and $(\mu(ds,d\xi)-\nu(s,d\xi)ds)\,du$. We thus obtain

$$\begin{aligned}
&\int_t^{T_n} r_u Z(t,u,x)\,du\\
&=\int_t^{T_n} r_u\Big(Z(0,u,x)+\int_0^t Z(s,u,x)\beta(s,u,x)\,dW_s\\
&\quad+\int_0^t Z(s-,u,x)\int_{(0,1]}\gamma(s,u,x,\xi)(\mu(ds,d\xi)-\nu(s,d\xi)ds)\Big)du\\
&=\int_0^{T_n} r_u Z(0,u,x)\,du\\
&\quad+\int_0^t \mathcal{B}(s,x)\,dW_s+\int_0^t\int_{(0,1]}\mathcal{C}(s,x,\xi)(\mu(ds,d\xi)-\nu(s,d\xi)ds)\\
&\quad-\int_0^t r_u Z(u,u,x)\,du,
\end{aligned}$$

which yields the claim. □

The gains process (16) accordingly simplifies to

$$\begin{aligned}
dG_t^{(x_1,x_2]}&=\int_{(x_1,x_2]}\Big\{s_0^{(x_1,x_2]}\sum_{t<T_i} dZ(t,T_i,x)+dZ(t,T_n,x)\\
&\quad+\int_t^{T_n} r_u Z(t,u,x)\beta(t,u,x)\,du\,dW_t\\
&\quad+\int_{(0,1]}\int_t^{T_n} r_u Z(t-,u,x)\gamma(t,u,x,\xi)\,du\,(\mu(dt,d\xi)-\nu(t,d\xi)dt)\Big\}dx\\
&=e^{-\int_0^t r_u\,du}\Big(B_t^{(x_1,x_2]}\,dW_t+\int_{(0,1]} C_t^{(x_1,x_2]}(\xi)\,(\mu(dt,d\xi)-\nu(t,d\xi)dt)\Big)
\end{aligned}$$

where

$$B_t^{(x_1,x_2]}=\int_{(x_1,x_2]}\Big\{s_0^{(x_1,x_2]}\sum_{t<T_i} P(t,T_i,x)\beta(t,T_i,x) \tag{19}$$
$$\quad+P(t,T_n,x)\beta(t,T_n,x)+\int_t^{T_n} r_u P(t,u,x)\beta(t,u,x)\,du\Big\}dx$$

$$C_t^{(x_1,x_2]}(\xi)=\int_{(x_1,x_2]}\Big\{s_0^{(x_1,x_2]}\sum_{t<T_i} P(t-,T_i,x)\gamma(t,T_i,x,\xi) \tag{20}$$
$$\quad+P(t-,T_n,x)\gamma(t,T_n,x,\xi)+\int_t^{T_n} r_u P(t-,u,x)\gamma(t,u,x,\xi)\,du\Big\}dx.$$

The predictable quadratic covariation thus computes to

$$\frac{d\langle G^{(x_1,x_2]},G^{(0,1]}\rangle}{dt}=e^{-2\int_0^t r_u\,du}\Big(B_t^{(x_1,x_2]}B_t^{(0,1]}-\int_{(0,1]} C_t^{(x_1,x_2]}(\xi)C_t^{(0,1]}(\xi)\,f(t,t,L_t+d\xi)\Big)$$

where we have used $\nu(t, d\xi) = -f(t, t, L_t + d\xi)$, which follows from (11) and (6). Hence the variance-minimizing strategy given by

$$\phi_t^* = \frac{B_t^{(x_1,x_2]} B_t^{(0,1]} - \int_{(0,1]} C_t^{(x_1,x_2]}(\xi) C_t^{(0,1]}(\xi)\, f(t, t, L_t + d\xi)}{(B_t^{(0,1]})^2 - \int_{(0,1]} (C_t^{(0,1]}(\xi))^2\, f(t, t, L_t + d\xi)} \tag{21}$$

can be computed at any time t by the observables

$$s_0^{(x_1,x_2]}, \quad P(t, u, x), \quad t \le u \le T_n, \quad x \in \mathcal{I},$$

and the model parameters

$$r_u, \quad \beta(t, u, x), \quad \gamma(t, u, x, \cdot), \quad t \le u \le T_n, \quad x \in \mathcal{I}.$$

The model parameters can be calibrated to the prevailing market data which could be either time series or option prices.

6 Affine Term Structure

In this section we consider a one-factor affine model proposed in Section 7.1 in [4]. This simple model is able to calibrate perfectly to any given initial term structure in the market and also allows for the explicit computation of the variance-minimizing hedging strategy as we show now.

Assume a constant risk-free short rate r. The factor Y is assumed to be a Feller square root process:

$$dY_t = (\mu_0 + \mu_1 Y_t)dt + \sigma\sqrt{Y_t}dW_t, \quad Y_0 = y \in \mathbb{R}_+ \tag{22}$$

and the forward rate follows an *affine term structure model*

$$f(t, T, x) = A'(t, T, x) + B'(t, T, x)Y_t$$

for some functions $A'(t, T, x)$ and $B'(t, T, x)$ with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote

$$A(t, T, x) = \int_t^T A'(t, u, x)du, \quad B(t, T, x) = \int_t^T B'(t, u, x)du.$$

The functions A and B are determined in terms of Riccati equations, which under (22) can be solved explicitly. From Section 7.1 in [4] we obtain that

$$\lambda(t, x) = \alpha_0(t, x) - r + \beta_0(x)Y_t, \tag{23}$$

with some $\mathbb{R}_+$-valued bounded measurable functions $\alpha_0(t, x)$ and $\beta_0(x)$ which are α_0 and β_0 are increasing and càdlàg, and $\alpha_0(t, 1) = r \ge 0$ and $\beta_0(1) = 0$.

This functions can be used to calibrated the model to an initial term structure of STCDO prices. The Riccati equations become

$$A(t,T,x) = \int_t^T \left(\alpha_0(s,x) + \mu_0 B(s,T,x)\right) ds \tag{24}$$

$$-\partial_t B(t,T,x) = \beta_0(x) + \mu_1 B(t,T,x) - \frac{\sigma^2}{2} B(t,T,x)^2, \quad B(T,T,x) = 0.$$

The equation for B has the solution

$$B(t,T,x) \equiv B(T-t,x) = \frac{2\beta_0(x)\left(e^{\rho(x)(T-t)} - 1\right)}{\rho(x)\left(e^{\rho(x)(T-t)} + 1\right) - \mu_1\left(e^{\rho(x)(T-t)} - 1\right)} \tag{25}$$

where $\rho(x) = \sqrt{\mu_1^2 + 2\sigma^2\beta_0(x)}$. Note that

$$A'(t,T,x) = \partial_T A(t,T,x) = \alpha_0(T,x) + \mu_0 B(T-t,x).$$

and therefore the forward rate takes the following form

$$f(t,T,x) = \alpha_0(T,x) + \mu_0 B(T-t,x) + \partial_T B(T-t,x) Y_t. \tag{26}$$

Moreover,

$$P(t,T,x) = 1_{\{L_t \leq x\}} e^{-A(t,T,x) - B(T-t,x)Y_t}.$$

In the following we compute the variance-minimizing hedging strategy in this model. We do not assume that the STCDO-prices are observed for any level x, but rather fix the attachment point structure $0 = x_0 < x_1 < \cdots < x_M = 1$. For simplicity we consider only $L_t = 0$.

In the following, we compute the essential terms for the hedging strategy. We assume that α_0 is piecewise linear and β_0 is piecewise constant:

(A3) Assume that $L_t = 0$ and

$$\alpha_0(s,x) = r + \sum_{i=1}^{M} (\alpha_{1,i}(s) + \alpha_{2,i}(s)x) 1_{\{x \in [x_{i-1}, x_i)\}},$$

$$\beta_0(x) = \sum_{i=1}^{M} \beta_i 1_{\{x \in [x_{i-1}, x_i)\}}.$$

Equation (23) shows that α_0 is the intensity of the loss process crossing level x when the factor Y equals zero. Under (A3) this is interpolated linearly, e.g. from the observed tranche prices. The factor β_i determines the (linear) influence of Y on the intensity.

Remark 6.1. *The assumption $L_t = 0$ is taken for simplicity of the notation. It is straightforward to extend the following results to the general case $L_t \geq 0$. By (23) and Remark 2.2, α_0 is decreasing in x with $\alpha_0(t,1) = r$ such that $\alpha_{2,i}(s) \leq 0$. Moreover, as $Y \geq 0$ also β_0 is decreasing in x and $\beta_0(1) = 0$.*

Hence $\beta_1 \geq \cdots \geq \beta_M \geq 0$. Continuity of α_0 in x eases the expressions at some places; continuity of α_0 in x is equivalent to

$$\alpha_{1,l+1}(t) + \alpha_{2,l+1}(t)x_l = \alpha_{1,l}(t)\alpha_{2,l}(t)x_l,$$

for all $t \geq 0$ and $0 \leq l \leq N$.

Under **(A3)**, we obtain that A is piecewise linear and B piecewise constant in x:

$$A(t,T,x) = \sum_{i=1}^{n} (A_{1,i}(t,T) + A_{2,i}(t,T)x) 1_{\{x \in [x_{i-1}, x_i)\}}$$
$$B(t,T,x) = \sum_{i=1}^{n} B_i(T-t) 1_{\{x \in [x_{i-1}, x_i)\}}$$

with

$$A_{1,i}(t,T) = \int_t^T (r + \alpha_{1,i}(s) + B_i(T-s))\, ds$$
$$A_{2,i}(t,T) = \int_t^T \alpha_{2,i}(s) ds$$
$$B_i(T) = \frac{2\beta_i \left(e^{\rho(i)T} - 1\right)}{\rho(i)\left(e^{\rho(i)T} + 1\right) - \mu_1 \left(e^{\rho(i)T} - 1\right)}$$

and $\rho(i) = \sqrt{\mu_1^2 + 2\sigma^2 \beta_i}$.

Lemma 6.2. *Assume that* **(A3)** *holds. Then, in the affine one-factor model we have*

$$B_t^{(x_{i-1},x_i]} = \sigma\sqrt{Y_t} \int_t^{T_n} \left(P(t,u,x_{i-1}) - P(t,u,x_i-)\right) \frac{B_i(u-t)}{A_{2,i}(t,u)} dw_u^{(x_{i-1},x_i]}$$

where

$$dw_u^{(x_{i-1},x_i]} = r\, du + \sum_{j=1}^{M} \left(s_0^{(x_{i-1},x_i]} + 1_{\{j=n\}}\right) \delta_{T_j}(du)$$

where δ_T is the Dirac measure at T.

Proof. From (26) and (22) we obtain that $b(t,T,x) = B'(T-t,x)\sigma\sqrt{Y_t}$. As $B(0,x) = 0$, inserting this in (4) gives

$$\beta(t,T,x) = -B(T-t,x)\sigma\sqrt{Y_t}.$$

Denote $w(i,j) := s_0^{(x_{i-1},x_i]} + 1_{\{j=n\}}$. Equation (19) yields together with $L_t = 0$

that

$$
\begin{aligned}
B_t^{(x_{i-1},x_i]} &= -\sigma\sqrt{Y_t}\int_{(x_{i-1},x_i]}\bigg\{s_0^{(x_{i-1},x_i]}\sum_{t<T_j}P(t,T_j,x)B(T_j-t,x)\\
&\qquad + P(t,T_n,x)B(T_n-t,x)+r\int_0^{T_n}P(t,u,x)B(u-t,x)\,du\bigg\}dx\\
&= -\sigma\sqrt{Y_t}\int_{(x_{i-1},x_i]}\bigg\{\sum_{t<T_j}w(i,j)P(t,T_j,x)B(T_j-t,x)\\
&\qquad + r\int_t^{T_n}P(t,u,x)B(u-t,x)\,du\bigg\}dx.
\end{aligned}
$$

The affine structure and **(A3)** allow to compute

$$
\begin{aligned}
\int_{(x_{i-1},x_i]}P(t,T,x)B(T-t,x)dx &= \int_{(x_{i-1},x_i)}e^{-A(t,T,x)-B(T-t,x)Y_t}B(T-t,x)dx\\
&= B_i(T-t)e^{-A_{1,i}(t,T)-B_i(T_t)Y_t}\int_{(x_{i-1},x_i)}e^{-A_{2,i}(t,T)x}dx\\
&= \frac{B_i(T-t)e^{-A_{1,i}(t,T)-B_i(T-t)Y_t}}{A_{2,i}(t,T)}\left(e^{-A_{2,i}(t,T)x_{i-1}}-e^{-A_{2,i}(t,T)x_i}\right)\\
&= \frac{B_i(T-t)}{A_{2,i}(t,T)}\left(P(t,T,x_{i-1})-P(t,T,x_i-)\right). \qquad (27)
\end{aligned}
$$

Hence

$$
\begin{aligned}
B_t^{(x_{i-1},x_i]} &= -\sigma\sqrt{Y_t}\bigg\{\sum_{t<T_j}w(i,j)\frac{B_i(T_j-t)}{A_{2,i}(t,T_j)}\left(P(t,T_j,x_{i-1})-P(t,T_j,x_i-)\right)\\
&\qquad + r\int_t^{T_n}\frac{B_i(u-t)}{A_{2,i}(t,u)}\left(P(t,u,x_{i-1})-P(t,u,x_i-)\right)du\bigg\},
\end{aligned}
$$

which is exactly the claim. □

The following result gives the remaining part of the hedging strategy. Let

$$
V(i,l,u) := \frac{x_l-x_{l-1}}{A_{2,i}(u)} - 1_{\{i=l\}}\frac{1}{(A_{2,i}(u))^2} \qquad (28)
$$

as well as $W(i,l,u) := V(i,l,u)$ for $1 \le i < l$ and $W(l,l,u) := -\frac{1}{(A_{2,l}(u))^2}$ and set

$$
w^k(i,u) := \sum_{l=(k+1)\wedge i}^{M}\alpha_{2,i}(t)W(i,l,u), \quad v^k(i,u) := \sum_{l=(k+1)\wedge i}^{M}\alpha_{2,i}(t)V(i,l,u). \qquad (29)
$$

Proposition 6.3. *Under* **(A3)** *we have that*

$$\int_{(0,1]} C_t^{(x_{k-1},x_k]}(\xi) C_t^{(0,1]}(\xi)\, f(t,t,L_t+d\xi)$$

$$= -C_t^{(x_{k-1},x_k]}(x_{k-1}) \cdot \left(\sum_{i=1}^{M} \int_t^{T_n} \big(v^k(i,u) P(t,u,x_i) - w^k(i,u) P(t,u,x_i-) \big) dv_u^{(0,1]} \right)$$

$$+ \alpha_{2,k}(t) I_k$$

$$+ \sum_{l=1}^{M} \Big(f(t,t,x_l) - f(t,t,x_l-) \Big) C_t^{(x_{k-1},x_k]}(x_l) C_t^{(0,1]}(x_l),$$

where I_k *is given in Equation* (32) *below and*

$$\begin{aligned} f(t,t,x_l) - f(t,t,x_l-) &= \alpha_{1,l+1}(t) - \alpha_{1,l}(t) + x_l(\alpha_{2,l+1}(t) - \alpha_{2,l}(t)) \\ &\quad + Y_t\big(B'_{l+1}(0) - B'_l(0)\big). \end{aligned}$$

If α_0 *is chosen to be continuous, then*

$$f(t,t,x_l) - f(t,t,x_l-) = Y_t\big(B'_{l+1}(0) - B'_l(0)\big).$$

Proof of Proposition 6.3

This section contains the proof of Proposition 6.3. For the proof we make use of the following result.

Lemma 6.4. *In the affine one-factor model under* **(A3)**,

$$C_t^{(x_{i-1},x_i]}(\xi) = -1_{\{\xi > x_{i-1}\}} \int_t^{T_n} \frac{P(t,u,x_{i-1}) - P(t,u,(x_i \wedge \xi)-)}{A_{2,i}(t,u)} dv_u^{(x_{i-1},x_i]}$$

and

$$C_t^{(0,1]}(\xi) = -\sum_{i=1}^{M} 1_{\{\xi > x_{i-1}\}} \int_t^{T_n} \frac{P(t,u,x_{i-1}) - P(t,u,(x_i \wedge \xi)-)}{A_{2,i}(t,u)} dv_u^{(0,1]}$$

with

$$v_u^{(x_1,x_2]} = \left(ru + \sum_{u<T_j} \Big(s_0^{(x_1,x_2]} + 1_{\{j=n\}} \Big) \right).$$

Proof. With $w(i,j) := s_0^{(x_{i-1},x_i]} + 1_{\{j=n\}}$ we obtain from Equation (20) that

$$\begin{aligned} C_t^{(x_{i-1},x_i]}(\xi) = \int_{(x_{i-1},x_i]} \Big\{ &\sum_{t<T_j} w(i,j) P(t-,T_j,x) \gamma(t,T_j,x,\xi) \\ &+ \int_t^{T_n} r P(t-,u,x) \gamma(t,u,x,\xi)\, du \Big\} dx. \end{aligned}$$

First, $c=0$ in (5) yields $\gamma(t,u,x,\xi)=-1_{\{L_{t-}+\xi>x\}}=-1_{\{\xi>x\}}$. Moreover, as in (27)

$$\int_{(x_{i-1},x_i]} P(t-,T,x)1_{\{\xi>x\}}dx = 1_{\{\xi>x_{i-1}\}}\frac{P(t,T,x_{i-1})-P(t,T,(x_i\wedge\xi)-)}{A_{2,i}(t,T)} \tag{30}$$

and we obtain

$$\begin{aligned} C_t^{(x_{i-1},x_i]}(\xi) = -1_{\{\xi>x_{i-1}\}}\Bigg\{ \sum_{t<T_j} w(i,j)\frac{P(t,T_j,x_{i-1})-P(t,T_j,(x_i\wedge\xi)-)}{A_{2,i}(t,T_j)} \\ + r\int_t^{T_n} \frac{P(t,u,x_{i-1})-P(t,u,(x_i\wedge\xi)-)}{A_{2,i}(t,u)}\,du\Bigg\}. \end{aligned} \tag{31}$$

The expression for $C_t^{(0,1]}(\xi)$ follows in a similar way. □

Proof of Proposition 6.3. Under (A3) f is piecewise linear but not necessarily continuous. With $\xi\in(0,1]$ we obtain

$$f(t,t,d\xi)=\sum_{l=1}^{M}\Big(1_{\{\xi\in[x_{l-1},x_l)\}}f_x(t,t,x_{l-1})d\xi+(f(t,t,x_l)-f(t,t,x_l-))\delta_{x_l}(d\xi)\Big),$$

where δ_x denotes the Dirac measure at x. We have that

$$\begin{aligned} f_x(t,t,x_{l-1}) &= \alpha_{2,l}(t), \\ f(t,t,x_l)-f(t,t,x_l-) &= \alpha_{1,l+1}(t)-\alpha_{1,l}(t)+x_l(\alpha_{2,l+1}(t)-\alpha_{2,l}(t)) \\ &\quad + Y_t\big(B'_{l+1}(0)-B'_l(0)\big). \end{aligned}$$

Next,

$$\begin{aligned} \int_{(0,1]} C_t^{(x_{k-1},x_k]}(\xi)C_t^{(0,1]}(\xi)f(t,t,d\xi) = \sum_{l=1}^{M}\alpha_{2,l}(t)\underbrace{\int_{[x_{l-1},x_l)} C_t^{(x_{k-1},x_k]}(\xi)C_t^{(0,1]}(\xi)\,d\xi}_{=:I_l} \\ + \sum_{l=1}^{M}\Big(f(t,t,x_l)-f(t,t,x_l-)\Big)C_t^{(x_{k-1},x_k]}(x_l)C_t^{(0,1]}(x_l) \end{aligned}$$

For $l<k$ the integral vanishes. For $l>k$ we have from (30) that

$$C_t^{(x_{k-1},x_k]}(\xi)=-\int_t^{T_n}\frac{P(t,u,x_{k-1})-P(t,u,x_k-)}{A_{2,k}(u)}dv_u^{(x_{k-1},x_k]}=C_t^{(x_{k-1},x_k]}(1)$$

and

$$\int_{(x_{l-1},x_l]} C_t^{(0,1]}(\xi)\,d\xi$$

$$= -\int_t^{T_n} \sum_{i=1}^{l} \int_{(x_{l-1},x_l]} \frac{P(t,u,x_{i-1}) - P(t,u,(x_i \wedge \xi)-)}{A_{2,i}(u)} d\xi dv_u^{(0,1]}$$

$$= -\sum_{i=1}^{l-1} (x_l - x_{l-1}) \int_t^{T_n} \frac{P(t,u,x_{i-1}) - P(t,u,x_i-)}{A_{2,i}(u)} dv_u^{(0,1]}$$

$$- \int_t^{T_n} \left\{ \frac{P(t,u,x_{l-1})}{A_{2,l}(u)} \Big((x_l - x_{l-1}) - (A_{2,l}(u))^{-1} \Big) + \frac{P(t,u,x_l-)}{(A_{2,l}(u))^2} \right\} dv_u^{(0,1]}$$

$$= -\sum_{i=1}^{l} \int_t^{T_n} \Big(V(i,l,u) P(t,u,x_{i-1}) - W(i,l,u) P(t,u,x_i-) \Big) dv_u^{(0,1]}.$$

with W, V given in (28). Hence

$$\sum_{l=k+1}^{M} I_l$$

$$= \sum_{l=k+1}^{M} \alpha_{2,l}(t) C_t^{(x_{k-1},x_k]}(1) \cdot \left(-\sum_{i=1}^{l} \int_t^{T_n} \Big(V(i,l,u) P(t,u,x_{i-1}) - W(i,l,u) P(t,u,x_i-) \Big) dv_u^{(0,1]} \right)$$

$$= -C_t^{(x_{k-1},x_k]}(1) \cdot \left(\sum_{i=1}^{M} \int_t^{T_n} \big(v^k(i,u) P(t,u,x_i) - w^k(i,u) P(t,u,x_i-) \big) dv_u^{(0,1]} \right)$$

where w, v is as in (29). Finally, we consider the case where $l = k$. Then

$$I_k$$

$$= \int_{x_{k-1}}^{x_k} \int_t^{T_n} \frac{P(t,u,x_{k-1}) - P(t,u,\xi)}{A_{2,k}(u)} \, dv_u^{(x_{k-1},x_k]} \cdot \int_t^{T_n} \sum_{i=1}^{k} \frac{P(t,z,x_{i-1}) - P(t,z,x_i \wedge \xi)}{A_{2,i}(z)} dv_z^{(0,1]} d\xi$$

$$= \int_t^{T_n} \frac{P(t,u,x_{k-1}) \big((A_{2,k}(u))(x_k - x_{k-1}) - 1 \big) + P(t,u,x_k-)}{(A_{2,k}(u))^2} dv_u^{(x_{k-1},x_k]}$$

$$\cdot \left(\sum_{i=1}^{k-1} \int_t^{T_n} \frac{P(t,z,x_{i-1}) - P(t,z,x_i-)}{A_{2,i}(z)} dv_z^{(0,1]} \right)$$

$$+ \int_t^{T_n} \int_t^{T_n} \int_{x_{k-1}}^{x_k} \frac{(P(t,u,x_{k-1}) - P(t,u,\xi))(P(t,z,x_{k-1}) - P(t,z,\xi))}{A_{2,k}(u) A_{2,k}(z)} d\xi \, dv_u^{(x_{k-1},x_k]} dv_z^{(0,1]}.$$

Note that

$$\sum_{i=1}^{k-1} \int_t^{T_n} \frac{P(t,u,x_{i-1}) - P(t,u,x_i)}{A_{2,i}(u)} dv_u^{(0,1]} = C_t^{(0,1]}(x_{k-1})$$

and

$$\int_{(x_{k-1},x_k]} \Big((P(t,u,x_{k-1}) - P(t,u,\xi))(P(t,z,x_{k-1}) - P(t,z,x_k))\Big)d\xi$$
$$= P(t,u,x_{k-1})P(t,z,x_{k-1})\left((x_k - x_{k-1}) - \frac{1}{A_{2,k}(u)} - \frac{1}{A_{2,k}(z)} + \frac{1}{A_{2,k}(u) + A_{2,k}(z)}\right)$$
$$+ P(t,u,x_{k-1})P(t,z,x_k-)\frac{1}{A_{2,k}(z)}$$
$$+ P(t,u,x_k-)P(t,z,x_{k-1})\frac{1}{A_{2,k}(u)}$$
$$- P(t,u,x_k-)P(t,z,x_k-)\frac{1}{A_{2,k}(u)A_{2,k}(z)}.$$

Summarizing,

$$I_k \tag{32}$$
$$= \int_t^{T_n} \frac{P(t,u,x_{k-1})\big((A_{2,k}(u))(x_k - x_{k-1}) - 1\big) + P(t,u,x_k-)}{(A_{2,k}(u))^2} dv_u^{(x_{k-1},x_k]} \cdot C_t^{(0,1]}(x_{k-1})$$
$$+ \int_t^{T_n}\int_t^{T_n} \frac{1}{A_{2,k}(u)A_{2,k}(z)}\Bigg\{P(t,u,x_{k-1})P(t,z,x_k-)\frac{1}{A_{2,k}(z)}$$
$$+ P(t,u,x_{k-1})P(t,z,x_{k-1})\left((x_k - x_{k-1}) - \frac{1}{A_{2,k}(u)} - \frac{1}{A_{2,k}(z)} + \frac{1}{A_{2,k}(u) + A_{2,k}(z)}\right)$$
$$+ P(t,u,x_k-)P(t,z,x_{k-1})\frac{1}{A_{2,k}(u)} - P(t,u,x_k-)P(t,z,x_k-)\frac{1}{A_{2,k}(u)A_{2,k}(z)}\Bigg\}dv_u^{(x_{k-1},x_k]}dv_z^{(0,1]}.$$

□

7 Conclusion and Outlook

This paper derives dynamic hedging strategies for a large class of top-down models for CDO markets. The goal is to hedge single-tranche CDOs with the CDO index. Explicit formulas are provided for a simple one-factor affine model. Further studies shall analyse the empirical performance of the model and the hedging strategies; of particular importance is the comparison to other approaches in the literature.

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