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Abstract

In this work the closed form Black-Scholes like formula for a European call option on a stock with no dividend is given where the interest rate is driven by Vasicek model. Mainly we utilize the change of numeraire technique firstly given in Jamshidian (1989), Geman et. al. (1995) *and reviewed in Björk (1998), Brigo and Mercurio (2001).*

Starting with the question ‘How the well known Black-Scholes formula would look like if the interest rates were stochastic with a Vasicek model‘ we found that there have been already given an answer to this question (see Rabinovitch (1989)). Rabinovitch used mainly the Merton‘s approach to drive a closed form formula for the price of a call option under the setting that the interest rate dynamics is driven by an O-U type process. Similarly, the objective of our study is to derive a closed form B-S like formula for a European call option on an underlying stock S, with date of maturity T and strike price K. We mainly utilize the change of numeraire technique which makes computations simpler. Moreover we calculate the sensitivities of the option price with respect to the model parameters. The result is slightly different than the one revealed in Rabinovitch (1989).

The standard assumptions of the Black-Scholes model except the constant interest rate are satisfied in our framework. We assumed that interest rates evolve according to the Vasicek model. In this setting the following dynamics for the stock price and interest rate processes are assumed to hold under the risk neutral measure Q

$$dS(t) = S(t)r(t)dt + S(t)\sigma_1 dW_t^1 \tag{1}$$

$$dr(t) = (b + \beta r(t))dt + \sigma_2 dW_t^2 \tag{2}$$

Here W^1 and W^2 are assumed to be two correlated Q-Wiener processes with correlation ρ. It is known that we can represent W_t^2 by using W_t^1 and W_t^3 for some W_t^1 and W_t^3 which are two independent Wiener processes. That is we have

$$dr(t) = (b + \beta r(t))dt + \sigma_2 \rho dW_t^1 + \sigma_2 \sqrt{1-\rho^2} dW_t^3 \tag{3}$$

As it is very well known that the fair price at time $t = 0$ of a contingent claim X is given by the formula

$$\Theta(0, X) = E^Q[\exp\{-\int_0^T r(s)ds\}X]. \tag{4}$$

where Q represents the risk neutral measure. Although the computation of the above formula is possible under the assumption of constant interest rate, it is quite difficult to come up with a closed form solution when interest rate has a stochastic dynamics. To handle with this problem a change of numeraire technique, in which the asset price processes are normalized by a numeraire asset price process, is used.

Recall that in the standard risk neutral pricing approach we mainly seek for a measure Q under which all the discounted price processes i.e., $\frac{\Theta(t)}{B(t)}$ are Q-martingales, where the bank account process $B(t)$ is used as a numeraire. However, it is shown in Jamshidian (1989) and Geman et. al.(1995) that with other choices of numeraires the martingale property still holds. That is, under certain conditions, for a tradable numeraire process $S_0(t)$ the existence of a measure Q^0 is guaranteed such that $\frac{\Theta(t)}{S_0(t)}$ is a Q^0- martingale for every asset price process $\Theta(t)$. Moreover it is stated that for every T- claim X

$$\Theta(t;X) = S_0(t)E_t^0[\frac{X}{S_0(T)}] \tag{5}$$

holds where E^0 represents the expectation with respect to Q^0. Here notice that, in formula (5) taking a T- zero bond price process $P(t,T)$ as the numeraire process we have the following equation

$$\Theta(t;X) = P(t,T)E_t^T[\frac{X}{P(T,T)}] \tag{6}$$

where E^T stands for the expectation under the measure Q^T which is known as the *forward measure*. Due to the well known fact that $P(T,T) = 1$, the formula (6) reduces to the following

$$\Theta(t;X) = P(t,T)E_t^T[X] \tag{7}$$

In what follows, to derive a closed form formula for a call option under the stochastic interest rates setting, we will mainly use this change of numeraire methodology given in Geman et. al. (1995) and Björk (1998).

Our problem is to find a closed form expression to the formula (4) where in the case of a call option pricing we have the payoff structure $X = \max[S(T) - K, 0]$. By using the indicator function $\mathcal{I}$ one can rewrite X as

$$X = [S(T) - K]\mathcal{I}_{\{S(T)\geq K\}}. \tag{8}$$

If we plug (8) in to the formula (4) we have

$$\Theta(0,X) = E^Q[\exp\{-\int_0^T r(s)ds\}[S(T)-K]\mathcal{I}_{\{S(T)\geq K\}}] \tag{9}$$

$$= \underbrace{E^Q[\exp\{-\int_0^T r(s)ds\}S(T)\mathcal{I}_{\{S(T)\geq K\}}]}_{I} - \underbrace{E^Q[\exp\{-\int_0^T r(s)ds\}K\mathcal{I}_{\{S(T)\geq K\}}]}_{II} \tag{10}$$

In equation (10), part I we take the stock price process $S(t)$ as a numeraire and by using equation (5) we write I in the following form

$$I = S(0)E_t^{Q^S}[\frac{S(T)\mathcal{I}_{\{S(T)\geq K\}}}{S(T)}] \tag{11}$$

which is equal to

$$I = S(0)Q^S(S(T)\geq K) \tag{12}$$

Similarly, in II using formula (6) by choosing $P(t,T)$ as a numeraire i.e., having the forward measure as the pricing measure, we have the following

$$II = P(0,T)E_t^{Q^T}[K\mathcal{I}_{\{S(T)\geq K\}}] \tag{13}$$

which becomes

$$II = P(0,T)KQ^T(S(T)\geq K) \tag{14}$$

It is clear from the equations (12) and (14) that if we can compute the probabilities in these equations, the closed form formula for the call option price will be directly obtained. To do this, by following the same methodology with Björk (1998) we introduce the process $Z(t) = \frac{S(t)}{P(t,T)}$ and in what follows we derive a closed form formula for a special case in which the interest rate evolves according to Vasicek model.

Let us define $Z(t) = \frac{S(t)}{P(t,T)}$ and write the probability in (II) as

$$Q^T(S(T)\geq K) = Q^T(\frac{S(T)}{P(T,T)}\geq K) = Q^T(Z(T)\geq K) \tag{15}$$

To calculate the probability $Q^T(Z(T)\geq K)$ we should have some knowledge about the dynamics of $Z(t)$. It is clear that $Z(t)$ has zero drift under Q^T since it is a normalized asset price process. In other words the Q^T dynamics of $Z(t)$ is in the form

$$dZ(t) = Z(t)\sigma_Z(t)dW^T \tag{16}$$

where W^T is a multidimensional Q^T-Wiener process. Indeed (16) is a Log-normal process and the solution is given by

$$Z(T) = Z(0) \exp\{-\frac{1}{2}\int_0^T ||\sigma_Z(t)||^2 dt + \int_0^T \sigma_Z(t) dW^T\} \tag{17}$$

If we have σ_Z deterministic then the stochastic integral in (17) has a normal distribution with mean zero and variance $\Sigma^2(T) = \int_0^T ||\sigma_Z(t)||^2 dt$. In Björk (1998) the following formula is given

$$Q^T(S(T) \geq K) = N(d_2) \tag{18}$$

where

$$d_2 = \frac{\ln(\frac{S(0)}{KP(0,t)}) - \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}} \tag{19}$$

Now the aim is to find the explicit form of σ_Z where the stock price and interest rate process is in the form (1) and (3) respectively.To find σ_Z we have to write the dynamics of $Z(t)$ explicitly. We can achieve this by using the integration by parts formula which states that

$$dZ(t) = d(\frac{S(t)}{P(t,T)}) = dS(t)\frac{1}{P(t,T)} + S(t)d(\frac{1}{P(t,T)}) + d\langle S, \frac{1}{P(t,T)}\rangle(t) \tag{20}$$

To apply (20) we need the dynamics of $S(t)$ and $\frac{1}{P(t,T)}$. We already know the Q-dynamics of $S(t)$ which is given in (1). To find the Q-dynamics of $\frac{1}{P(t,T)}$ we can use the fact that Vasicek model is an affine model and zero coupon bond price admits a representation of the following form

$$P(t,T) = \exp\{-A(t,T) - B(t,T)r(t)\} \tag{21}$$

where

$$B(t,T) = \frac{1}{\beta}(e^{\beta(T-t)} - 1) \tag{22}$$

and

$$A(t,T) = \frac{\sigma_2(4e^{\beta(T-t)} - e^{2\beta(T-t)} - 2\beta(T-t) - 3)}{4\beta^3} + b\frac{e^{\beta(T-t)} - 1 - \beta(T-t)}{\beta^2} \tag{23}$$

In the Vasicek model$-b/\beta$ represents the mean reversion level whereas $-\beta$ stands as the speed of adjustment parameter. Here notice that the coefficient β is in general negative.

To find the dynamics of $P(t,T)$ we apply the Itô formula to the function $F(t,r(t)) = e^{A(t,T)-B(t,T)r(t)}$ and get

$$dF(t,r(t)) = (-A_t(t,T) - B_t(t,T)r(t))F(t,r(t))dt - F(t,r(t))B(t,T)dr(t)$$
$$+\frac{1}{2}F(t,r(t))B^2(t,r(t))d<r>_t \tag{24}$$

where $A_t(t,T)$ and $B_t(t,T)$ indicate the derivatives of A and B with respect to t and β and b are the parameters of the Vasicek model.

By plugging (3) in (24), we have the following Q-dynamics for $P(t,T)$

$$dP(t,T) = P(t,T)[(-A_t(t,T) - B_t(t,T)r(t) + \frac{1}{2}B^2(t,T)\sigma_2^2 - B(t,T)b - B(t,T)\beta r(t))dt$$
$$-B(t,T)\sigma_2\rho dW_t^1 + \sigma_2\sqrt{1-\rho^2}dW_t^3] \tag{25}$$

Here notice that the drift term of (25) reduces to $r(t)$ due to the fact that the term structure equation, that is

$$-A_t(t,T) - (1 + B_t(t,T)r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T)) = 0$$

must be satisfied, where in our case we have $\sigma(t,r) = \sigma_2$ and $\mu(t,r) = (b + \beta r(t))$. As a result we have the following

$$dP(t,T) = P(t,T)[r(t)dt - B(t,T)\sigma_2(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^3)]$$

Dynamics of the process $\frac{1}{P(t,T)}$ is derived by simply applying the Itô formula. The result is as follows

$$d(\frac{1}{P(t,T)}) = \frac{1}{P(t,T)}[(B^2(t,T)\sigma_2^2 - r(t))dt + B(t,T)\sigma_2(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^3)] \tag{26}$$

Now by using (1), (20) and (26) we can find the dynamics of $Z(t)$ as

$$dZ(t) = Z(t)[(B^2(t,T)\sigma_2^2 + B(t,T)\rho\sigma_1\sigma_2)dt + (\sigma_1 + B(t,T)\sigma_2\rho)dW_t^1$$
$$+B(t,T)\sigma_2\sqrt{1-\rho^2}dW_t^3] \tag{27}$$

A very well known property of the volatility process is that it is not affected by measure changes. Therefore in the Q^T dynamics of $Z(t)$, i.e., in (16) we have

$$\sigma_Z(t) = \Big(\ \sigma_1 + B(t,T)\sigma_2\rho, \quad B(t,T)\sigma_2\sqrt{1-\rho^2} \ \Big)$$

As a result the variance of $Z(T)$ becomes

$$\Sigma^2_\rho(T) = \int_0^T (\sigma_1^2 + 2B(t,T)\sigma_1\sigma_2\rho + B^2(t,T)\sigma_2^2)dt$$

If we plug (22) in to the formula above we have

$$\Sigma^2(T) = T\sigma_1^2 + \frac{\sigma_2^2}{\beta^3}(\beta T + \frac{3}{2} - 2e^{\beta T} + \frac{1}{2}e^{2\beta T}) + \frac{2\sigma_1\sigma_2\rho}{\beta^2}(e^{\beta T} - \beta T - 1) \tag{28}$$

Using (28) in formula (19) we end up with a closed form expression for $Q^T(S(T) \geq K)$.

The next task is to calculate the probability which appears in the I part of (10). To do this the process $Y(t) = \frac{P(t,T)}{S(t)}$ is introduced. Now let us rewrite the probability $Q^S(S(T) \geq K)$ in the following way

$$Q^S(S(T) \geq K) = Q^S(\frac{P(T,T)}{S(T)} \leq \frac{1}{K}) = Q^S(Y(T) \leq \frac{1}{K})$$

Notice that under Q^S, $Y(t)$ has a zero drift and than we have the following expression

$$dY(t) = Y(t)\sigma_Y(t)dW^S$$

where W^S is a multidimensional Q^S-Wiener process. To calculate the probability $Q^S(Y(T) \leq \frac{1}{K})$ we need the expression of σ_Y.

Since we have $Y(t) = \frac{1}{Z(t)}$ by applying the Itô formula we get $\sigma_Y = -\sigma_Z$. That is we have the following expression

$$\sigma_Y(t) = \Big(\ -\sigma_1 - B(t,T)\sigma_2\rho, \quad -B(t,T)\sigma_2\sqrt{1-\rho^2} \ \Big)$$

As a result we have the following representation for $Y(T)$

$$Y(T) = Y(0)\exp\{-\frac{1}{2}\int_0^T ||\sigma_Z(t)||^2 dt - \int_0^T \sigma_Z(t)dW^S\}$$

By using the results above it can be shown that

$$Q^S(S(T) \geq K) = N(d_1)$$

where

$$d_1 = d_2 + \sqrt{\Sigma^2(T)}$$

and d_2 and Σ^2 are as in (19) and (28) respectively.

The results up to here reveal that under the assumption of Vasicek interest rates, price of a call option written on a stock S, with maturity T and strike price K is given by the following formula

$$\Theta(0) = S(0)N(d_1) - KP(0,T)N(d_2) \tag{29}$$

where we have

$$d_2 = \frac{\ln(\frac{S(0)}{KP(0,t)}) - \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}}$$

$$d_1 = d_2 + \sqrt{\Sigma^2(T)}$$

and

$$\Sigma^2(T) = T\sigma_1^2 + \frac{\sigma_2^2}{\beta^3}(\beta T + \frac{3}{2} - 2e^{\beta T} + \frac{1}{2}e^{2\beta T}) + \frac{2\sigma_1\sigma_2\rho}{\beta^2}(e^{\beta T} - \beta T - 1)$$

Our results differs from the Rabinovitch‘s (see Rabinovitch (1989), equation (8)) in the sense that in the expression of $\Sigma(T)$ we have a plus sign in front of the last term, that is, the term with the coefficient ρ.

After deriving the call option price we next calculate the sensitivities of the call price with respect to the model variables.

Sensitivity of Θ w.r.t Σ^2

$$\frac{\partial\Theta}{\partial\Sigma^2} = S(0)\frac{\partial N}{\partial d_1}\frac{\partial d_1}{\partial\Sigma^2} - KP(0,T)\frac{\partial N}{\partial d_2}\frac{\partial d_2}{\partial\Sigma^2}$$

$$= S(0)\phi(d_1)\frac{\sqrt{\Sigma^2}/2 - (1/2\sqrt{\Sigma^2})(\ln(S/K) - \ln P(0,T) + \Sigma^2/2)}{\Sigma^2}$$

$$-KP(0,T)\phi(d_2)\frac{-\sqrt{\Sigma^2}/2 - (1/2\sqrt{\Sigma^2})(\ln(S/K) - \ln P(0,T) - \Sigma^2/2)}{\Sigma^2}$$

Here $\phi(.)$ indicates the standard normal pdf. We have $\phi(d_2) = \phi(d_1)\frac{S_0}{KP(0,T)}$. Under this substitution we obtain

$$\frac{\partial\Theta}{\partial\Sigma^2} = \frac{S(0)\phi(d_1)}{\Sigma^2}(\sqrt{\Sigma^2}/2 - \frac{1}{2\sqrt{\Sigma^2}}(\ln(S/KP(0,T) + \Sigma^2/2) + \sqrt{\Sigma^2}/2$$
$$+\frac{1}{2\sqrt{\Sigma^2}}(\ln(S/KP(0,T) + \Sigma^2/2))$$

After making the necessary cancelations we derive the following expression

$$\frac{\partial\Theta}{\partial\Sigma^2} = \frac{S(0)\phi(d_1)}{\sqrt{\Sigma^2}}$$

Here notice that the sensitivity of the option price is positive with respect to Σ^2 which means that the option price increases with the volatility as the finance theory suggests.

Sensitivity of Θ w.r.t $P(0,T)$

$$\frac{\partial\Theta}{\partial P(0,T)} = S(0)\frac{\partial N}{\partial d_1}\frac{\partial d_1}{\partial P(0,T)} - KN(d_1) - KP(0,T)\frac{\partial N}{\partial d_2}\frac{\partial d_2}{\partial P(0,T)}$$

$$= -S(0)\phi(d_1)\frac{1}{P(0,T)\sqrt{\Sigma^2}} - KN(d_2) + K\phi(d_2)\frac{1}{\sqrt{\Sigma^2}}$$

Here we again use the fact that $\phi(d_2) = \phi(d_1)\frac{S_0}{KP(0,T)}$ and we get

$$= \frac{-S(0)\phi(d_1)}{P(0,T)\sqrt{\Sigma^2}} - KN(d_2) + \frac{S(0)\phi(d_1)}{P(0,T)\sqrt{\Sigma^2}})$$

which becomes

$$\frac{\partial\Theta}{\partial P(0,T)} = -KN(d_2)$$

The result shows us that the partial derivative of the call price with respect to the bond price is negative. One can explain this by using the relation between option prices and interest rates. We know that price of a call option increases with the interest rates since high rates has an effect on investors such that investors delay their purchases of stocks to a future date and invest more on bonds today. To guarantee the future acquisition of the stock, investors increase their demand for call options which causes an increase of the call option price. Under the light of this explanation to see the relation between bond prices and call option price, one can use the fact that the relation between bond prices and interest rates are negative.

Sensitivity of Θ w.r.t b

$$\frac{\partial\Theta}{\partial b} = \frac{\partial\Theta}{\partial\Sigma^2}\frac{\partial\Sigma^2}{\partial b} + \frac{\partial\Theta}{\partial P}\frac{\partial P}{\partial b}$$

We have $\frac{\partial\Sigma^2}{\partial b} = 0$
and

$$\frac{\partial P}{\partial b} = -P\frac{B(0,T) - T}{\beta}$$

Thus we have

$$\frac{\partial \Theta}{\partial b} = \frac{KPN(d_2)(B(0,T) - T)}{\beta} \tag{30}$$

Under the assumption that $\beta < 0$, (30) implies that the relation between Θ and b is positive. Recall that $-b/\beta$ is defined as the mean reversion level of the interest rates. Since we know that the direction of the relation between r and Θ is a positive and $-b/\beta$ is the long run level of the interest rate r , for fixed values of $-\beta$ it is not surprising to find a positive relation between Θ and b. To illustrate this statement, by fixing all other parameters we plot the values of the option price for different mean reversion levels $-b/\beta$. Results are given in *Figure 1*.

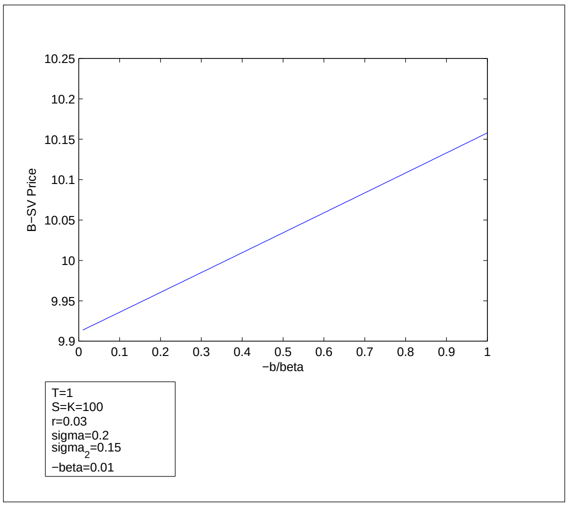

Figure 1: b vs B-SV Price

Sensitivity of Θ w.r.t β

When we calculate the sensitivity of the call option price with respect to minus the speed of the adjustment parameter, i.e, β, we realized that it is a little bit complicated and the sign of that sensitivity value changes according to the value of the other parameters, mainly according to b and σ_2. Intuitively the relation between b and β makes sense since $-\beta$ is the speed that the interest rate reaches to the mean reversion level and the relation between interest rates and the option price is positive. That is, if the mean reversion level is above (below) the current interest rate and if interest rate adjusts to this high (low) level quickly such a case results in an increase (decrease) in the option price. To show this relation we plotted the relation between the speed of adjustment, namely $-\beta$ and the option price for the cases where the mean reversion level is above and below the current interest rate r. Results are given in *Figure 2a* and *Figure 2b* Secondly, for changing values of σ_2 we figured

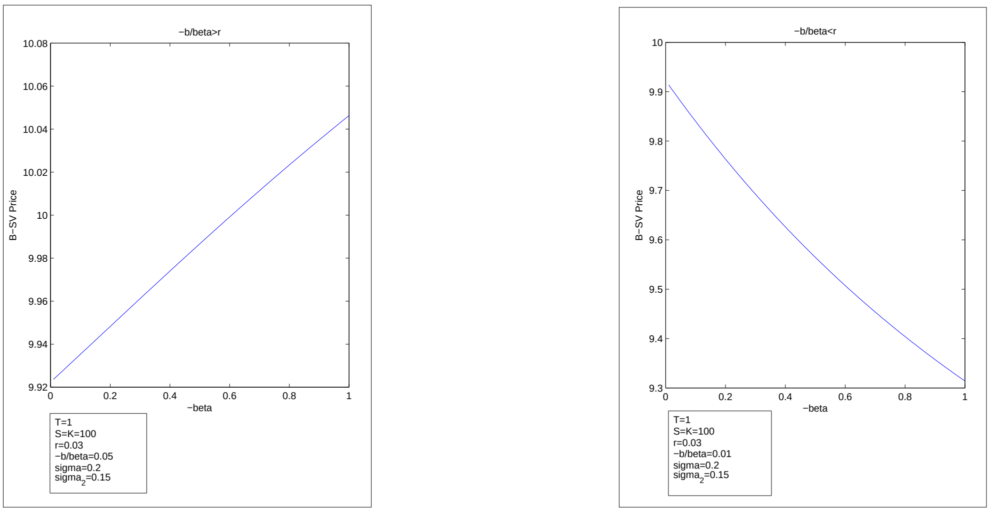

Figure 2: a)$-\beta$ vs B-SV Price when $-b/\beta > r$ b)$-\beta$ vs B-SV Price when $-b/\beta < r$

out the relation between $-\beta$ and the call option price. Results are in *Figure 3*.

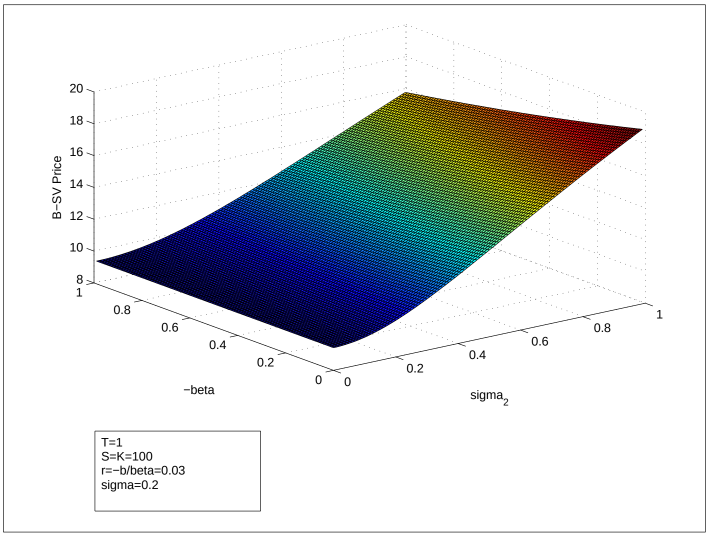

Figure 3: $-\beta$ vs B-SV Price for changing values of σ_2

Our next aim is to find a systematic relation between the call prices implied by the Black-Scholes (B-S) and Black-Scholes with Vasicek driven interest rate (B-SV) models. To do this we investigate the relation between the volatilities of the two models, namely the relation between $T\sigma_1^2$ and $\Sigma^2(T)$. Remember that $\Sigma^2(T)$, the volatility of the process Z, is given in (28) as the sum of $T\sigma_1^2$ and an additional part. In the additional part the first term is positive as long as β is negative. However in the second term the sign changes according to the value of ρ. To see the direction of the relation between Σ^2 and ρ we calculate the partial derivative of the former with respect to the later. The result is as follows

$$\frac{\partial \Sigma^2}{\partial \rho} = 2\sigma_1\sigma_2 \int_0^T B(t,T)dt \tag{31}$$

Under the assumption that $\beta < 0$ the expression (31) is always positive which implies that the variance of $Z(T)$ increases with ρ. Indeed it is naturel to end up with such a result. Intuitively speaking, for values of ρ which are close to -1, that is, a situation where the stock price process and interest rates move in the opposite directions, the stock and the bond will move as if they were the same assets. This implies less uncertainty in the market, thus a smaller variance. This result together with the result on the positivity of the relation between B-SV and Σ^2 yields that B-SV option price increases together with ρ. To see the relation between changing values of ρ and the difference between B-S and

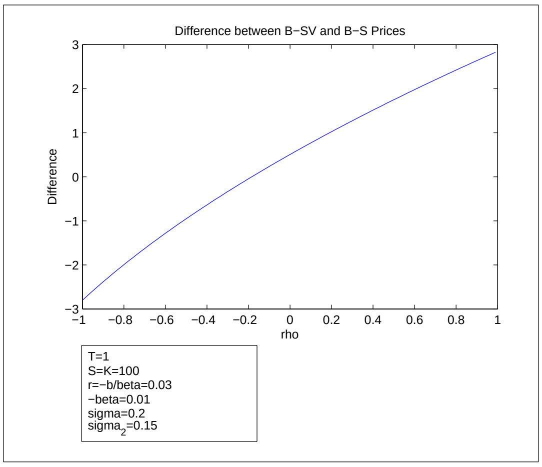

Figure 4: Difference between B-SV and B-S call prices for different values of ρ

B-SV prices we simulate the difference for different values of ρ by taking the values of the other parameters fixed. The result is given in *Figure 4* where the the difference between

B-SV and B-S option prices increases with the increasing value of the ρ. This result is compatible with the previous findings.

Assuming that $\rho = 0$, we investigate for the conditions where B-S option price is equal to B-SV price. The equality is possible when $\sigma_2 = 0$ and $r = -b/\beta$ simultaneously, meaning that the interest rate is on the long run equilibrium level. In such a case we have $P(0,T) = e^{-rT}$ and $\Sigma(T) = T\sigma_1^2$.

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