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A generalization of matrix Riccati differential equations: Existence and comparison results

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A GENERALIZATION OF MATRIX RICCATI DIFFERENTIAL EQUATIONS: EXISTENCE AND COMPARISON RESULTS

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Abstract. We introduce a generalization of matrix Riccati differential equations on the cone of positive semidefinite matrices, for which we derive global existence results under certain parameter restrictions. This generalization comprises for example all linear matrix differential equations with analytic coefficients, which leave the cone invariant. The standard matrix Lyapunov and matrix Riccati differential equations with analytic coefficients are naturally covered by this setting. We also supply standard comparison theorems for the respective differential inequalities. The results are compared to those achieved through standard comparison by Volkmann, Uhl et al. on the one hand, and to classic matrix Riccati differential equations, on the other hand.

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1. Introduction

The theory of symmetric and Hermitian matrix Riccati differential equations is well developed, due to their longstanding importance in different fields of mathematics and applied sciences. They occur, for instance, in optimal control, filtering theory and perturbation theory as well as differential geometry and have therefore motivated many generalizations into different directions (see [1] and the references therein). However, our motivation to generalize matrix Riccati equations is new. In joint work with D. Filipović and J. Teichmann ([3]) we provide a complete characterization of continuous affine Markov processes $(X_t^x)_{t\geq 0}$ on S_d^+, the cone of symmetric positive semidefinite matrices. By the very definition of such processes, their Laplace transform is exponentially affine in the state-variable x. That is, for $t \geq 0$ and $u \in S_d^+$, there exist $\phi(t,u) \in \mathbb{R}_+$ and $\psi(t,u) \in S_d^+$, such that for all $x \in S_d^+$

$$\mathcal{L}(X_t^x)(u) = e^{-\phi(t,u)-\langle x,\psi(t,u)\rangle}. \tag{1.1}$$

The functions $\phi(t,u)$ and $\psi(t,u)$ satisfy a system of ordinary differential equations of the form

$$\dot\phi(t,u) = \langle b, \psi(t,u)\rangle + c, \qquad \phi(0,u)=0, \tag{1.2}$$

$$\dot\psi(t,u) = -\psi(t,u)\alpha\psi(t,u) + \beta(\psi(t,u)) + \gamma, \qquad \psi(0,u) = u \in S_d^+, \tag{1.3}$$

where b, α, $\gamma \in S_d^+$, $c \in \mathbb{R}_+$ and β is a linear map on the space of symmetric matrices S_d, satisfying some inward pointing drift condition to guarantee the existence of the semi-flow $\psi(t,u)$ in the cone S_d^+ for all times $t \geq 0$ (for details, see the following sections and [3]). Such linear functions β comprise the class of maps

$$u \mapsto \beta(u) = \beta_0 u + u\beta_0^T, \quad \beta_0 \in M_d(\mathbb{R}), \tag{1.4}$$

where $M_d(\mathbb{R})$ denotes the space of real-valued $d\times d$ matrices. In this case, equation (1.3) simplifies to a classical symmetric matrix Riccati differential equation with constant coefficients. That is

$$\dot\psi(t,u) = -\psi(t,u)\alpha\psi(t,u) + \beta_0\psi(t,u) + \psi(t,u)\beta_0^T + \gamma, \quad \psi(0,u) = u. \tag{1.5}$$

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Since far more linear maps β outside the class of form (1.4) can be considered, this article aims at establishing comparison and existence results for matrix Riccati equations of type (1.3) with some general linear map β.

In the case of classical matrix Riccati differential equations of type (1.5), comparison arguments rely on the associated Lyapunov differential equation with $\gamma = 0$ of the form[1]

$$\dot{X}(t) = \beta_0 X(t) + X(t)\beta_0^T, \quad X(0) = u. \tag{1.6}$$

Its (unique) solution is given by $X(t) = X(t, u) = e^{\beta_0 t} u e^{\beta_0^T t}$. This can be seen by setting $Y(t) := e^{-\beta_0 t} X(t) e^{-\beta_0^T t}$, since we then have

$$\begin{aligned}\dot{Y}(t) &= -\beta_0 e^{-\beta_0 t} X(t) e^{-\beta_0^T t} + e^{-\beta_0 t} \dot{X}(t) e^{-\beta_0^T t} - e^{-\beta_0 t} X(t) \beta_0^T e^{-\beta_0^T t} \\ &= e^{-\beta_0 t}\left(\dot{X}(t) - \beta_0 X(t) - X(t)\beta_0^T(t)\right) e^{-\beta_0^T t} = 0, \\ Y(0) &= u.\end{aligned}$$

Hence, $Y(t) = u$ and $X(t) = e^{\beta_0 t} u e^{\beta_0^T t}$ for all $t \in \mathbb{R}$. In particular, as $u \mapsto e^{\beta_0 t} u e^{\beta_0^T t}$ is an automorphism of S_d^+ and as the partial order $\preceq$ induced by the latter is preserved under automorphisms, we have $u \preceq v$ implies $X(t, u) \preceq X(t, v)$ for all $t \in \mathbb{R}$. As a consequence, if u is strictly positive, then $X(t, u)$ is strictly positive definite, for all $t \in \mathbb{R}$. (This can, of course, also be checked by calculating the determinant of $X(t)$). As equations of type (1.5) can be dominated (with respect to the partial order) by a Lyapunov differential equation of form (1.6), the statements carry over to the classical matrix Riccati differential equations. Furthermore, the same assertions also hold true in the time-dependent case by means of transition matrices for the associated differential equation $\dot{x}(t) = \beta_0(t)x(t)$ on $\mathbb{R}^d$.

In the case of generalized matrix Riccati differential equations of the form

$$\dot{X}(t) = -X(t)\alpha(t)X(t) + \beta(t)(X(t)) + \gamma(t), \quad X(0) = u, \tag{1.7}$$

where $\alpha(t), \gamma(t) \in S_d^+$ and $\beta(t) \in \mathcal{L}(S_d)$ (the space of linear maps on S_d) are continuous functions in t, we cannot proceed in the same straightforward manner as described above. In addition, a theorem of A.N. Stokes ([6], see also chapter 4 of [1]), asserts that in general (1.7) does not yield order preserving local flows, or in other words: Trajectories $X(t, u)$ will, for at least some data, eventually leave the cone for negative times $t < 0$. Hence the formulation of comparison has to be restricted to semi-flows $X(t, u)$, $t \geq 0$.

Nevertheless, we shall adopt the same strategy as put forward in [1], by first establishing differential inequalities with respect to the Lyapunov equation in section 3, and then applying these to matrix Riccati differential equations (see section 4). Section 2 addresses the problem of invariance of S_d^+ with respect to (1.7).

2. Preliminaries: Invariance of the cone of positive semidefinite matrices under Riccati differential equations

For $d \geq 2$, we denote by S_d the linear space of symmetric matrices with scalar product $\langle u, v\rangle := \operatorname{tr}(uv)$. S_d^+ stands for the cone of symmetric positive semidefinite matrices and S_d^{++} for its interior, the cone of strictly positive definite matrices. The boundary is denoted by $\partial S_d^+ = S_d^+ \setminus S_d^{++}$. Further, we set $\mathbb{R}_+ := [0, \infty)$. Both cones, S_d^+ and S_d^{++} induce order relations. We shall write for the partial order induced by S_d^+, $\preceq$, that is $x \preceq y$ if and only if $y - x \in S_d^+$. Similarly, S_d^{++} induces the strict (clearly non-reflexive) order $\prec$. For $u \in S_d^+$ (or S_d^{++}) we shall likewise write $u \succeq 0$ ($u \succ 0$). Finally, $\mathcal{L}(S_d)$ denotes the space of linear maps on S_d and $O(d)$ the orthogonal group of dimension d over $\mathbb{R}$.

Consider now the symmetric matrix differential equation

$$\dot{X}(t) = R(t, X(t)), \quad X(0) = u, \tag{2.1}$$

where the map $\mathbb{R}_+ \times S_d \ni (t, X) \mapsto R(t, X) \in S_d$ is assumed to be continuous in t and Lipschitz continuous in X. Under these assumptions (2.1) admits a unique local solution. We are interested

[1] In the sequel, we shall always write $X(t)$ for the solution of the matrix differential equation.

in solutions which remain in S_d^+ and recall therefore the concept of invariance for some subset $D \subset S_d$.

Definition 2.1. A non empty closed set D is called *invariant* under (2.1) if, for all $u \in D$ and $t \geq 0$, $X(t,u) \in D$ (as long as the solution exists).

As we shall state the conditions guaranteeing invariance in terms of normal vectors, we introduce the notion of the *normal cone.* If the the subset D is convex (which is the case for S_d^+), the normal cone to D at u_0 is given by

$$N_D(u_0) = \{v \in S_d | \langle v, u - u_0 \rangle \geq 0, \text{ for all } u \in D\}, \tag{2.2}$$

(see [5], Theorem 6.9 for example). As we consider inward pointing normal vectors, this corresponds except for a change of the sign to the definition of the normal cone given in [5]. We recall now well-known conditions guaranteeing the invariance of a subset D (see W. Walter [9], for instance).

Theorem 2.2. *Let $R(t,X)$ be as in* (2.1)*. Then a non empty closed set D is invariant under* (2.1) *if for all $u_0 \in \partial D$ and $v \in N_D(u_0)$*

$$\langle v, R(t,u_0) \rangle \geq 0, \quad \text{for } t \geq 0, \tag{2.3}$$

is satisfied.

In the case of S_d^+, the normal cone (2.2) at any point $u_0 \in \partial S_d^+$ can be described by the following lemma.

Lemma 2.3. *For $u_0 \in \partial S_d^+$, the normal cone* (2.2) *is given by*

$$N_{S_d^+}(u_0) = \{v \in S_d^+ | \langle v, u_0 \rangle = 0\}, \quad u_0 \in \partial S_d^+. \tag{2.4}$$

Proof. $\supseteq$: Let $v \in S_d^+$ with $\langle v, u_0 \rangle = 0$. Then, $\langle v, u - u_0 \rangle = \langle v, u \rangle \geq 0$ for all $u \in S_d^+$.
$\subseteq$: We first claim that $v \in N_{S_d^+}(u_0)$ implies $\langle v, u_0 \rangle = 0$. To this end, we assume first that u_0 is diagonal and of the form $u_0 = \operatorname{diag}(\lambda_1 > 0, \ldots, \lambda_r > 0, 0, \ldots, 0)$. Assume, by contradiction that $\langle v, u_0 \rangle \neq 0$. Then, $v_{jj} \neq 0$ for some $j \in \{1, \ldots, r\}$. Take now

$$u = \operatorname{diag}(\lambda_1, \ldots, \lambda_j - \operatorname{sgn}(v_{jj})\frac{\lambda_j}{2}, \ldots, \lambda_r, 0 \ldots, 0),$$

which clearly lies in S_d^+. But then

$$\langle v, u - u_0 \rangle = \frac{-|v_{jj}|\lambda_j}{2} < 0.$$

Thus, $v \notin N_{S_d^+}(u_0)$, which contradicts our assumption. If u_0 is not diagonal, one can reduce to the first case by diagonalizing. Indeed, we have $0 \leq \langle v, u - u_0 \rangle = \langle UvU^T, UuU^T - Uu_0U^T \rangle := \langle \bar{v}, \bar{u} - \bar{u}_0 \rangle$ with diagonal $\bar{u}_0$ for some $U \in O(d)$. This implies $\bar{v} \in N_{S_d^+}(\bar{u}_0)$ and we can proceed as above, which yields $0 = \langle \bar{v}, \bar{u}_0 \rangle = \langle v, u_0 \rangle$.
Finally, due to the self-duality of S_d^+, $\langle v, u - u_0 \rangle = \langle v, u \rangle \geq 0$ for all $u \in S_d^+$ implies $v \in S_d^+$. □

Remark 2.4. Notice that the set of zero divisors of u_0 lying in S_d^+, i.e. $vu_0 = u_0 v = 0$ for $v \in S_d^+$, is equivalent to the normal cone at u_0.

An application of Theorem 2.2 to functions $R(t,X)$ of the form

$$R(t,X) = -X\alpha(t)X + \beta(t)(X) + \gamma(t), \tag{2.5}$$

where for each $t \geq 0$, $(\alpha(t), \beta(t), \gamma(t)) \in S_d \times \mathcal{L}(S_d) \times S_d$, allows to find conditions on the parameters such that S_d^+ is invariant under $\dot{X}(t) = R(t, X(t))$. This gives rise to the following definition.

Definition 2.5. (Admissibility) The parameters $(\alpha, \beta, \gamma) \in S_d \times \mathcal{L}(S_d) \times S_d$ are called *admissible*, if α, $\gamma \in S_d^+$ and if β satisfies

$$(\forall u_0 \in \partial S_d^+)(\forall v \in N_{S_d^+}(u_0))\left(\langle \beta(u_0), v \rangle \geq 0\right). \tag{2.6}$$

Furthermore, we write the map $u \mapsto \beta(u)$ in coordinates as $\beta(u)_{ij} = \sum_{1 \leq k \leq l \leq d} {\beta_{ij}}^{kl} u_{kl}$.

Note that β is admissible if and only if β^T is. We remark that through Theorem 2.2 no condition on α is directly imposed. We only require the positive semidefinitess in view of global existence (see section 4).
In the following observations we shall write $I\backslash\{k\}$ for $\{1,\dots,d\}\backslash\{k\}$. If $u \in S_d$, then $u_{I\backslash\{k\}}$ denotes the matrix where the k^{th} line and column is deleted.

Remark 2.6. Suppose $R(t,X)$ is of form (2.5). Then the following statements hold:

(1) S_d^+ is invariant under (2.1) if (α,β,γ) are admissible parameters.
(2) The set of all admissible parameters is a convex cone in $S_d\times\mathcal{L}(S_d)\times S_d$.
(3) For any admissible β we have ${\beta_{I\backslash\{k\}}}^{kk}\in S_{d-1}^+$ for $k=1,\dots,d$. This implies in particular ${\beta_{ii}}^{kk}\geq 0$ for all $1\leq i,k\leq d$, $i\neq k$.

Proof. Assertion (1) is clear by Theorem 2.2 and the second one is simply a consequence of the fact that S_d^+ is a convex cone and that $\langle\beta(u_0),v\rangle$ remains nonnegative when multiplying with a positive scalar.
In order to prove (3), we insert $u_0=e^{kk}$ in condition (2.6), where the $(ij)^{th}$ entry of e^{kk} is given by ${e_{ij}}^{kk}=\delta_{ik}\delta_{jk}$ with δ_{ij} the Kronecker delta. Thus,

$$\langle\beta(e^{kk}),v\rangle\geq 0 \text{ for all } v\in N_{S_d^+}(e^{kk}).$$

Since the k^{th} column and line of every element of $N_{S_d^+}(e^{kk})$ is zero (see also Remark 2.4), we have

$$\langle\beta(e^{kk}),v\rangle=\langle\beta^{kk},v\rangle=\left\langle{\beta_{I\backslash\{k\}}}^{kk},v_{I\backslash\{k\}}\right\rangle\geq 0.$$

This must hold true for all $v_{I\backslash\{k\}}\in S_{d-1}^+$, whence ${\beta_{I\backslash\{k\}}}^{kk}\in S_{d-1}^+$. □

3. Lyapunov equation and comparison statements.

In order to establish comparison results for equations of type (1.7), we shall employ the following theorem concerning regular choices of eigenvalues and eigenvectors along curves of symmetric operators (T. Kato, [4], II. §6. 2, see also [2]):

Theorem 3.1. *Let $A(t)$ be a real analytic curve of symmetric matrices. Then the eigenvalues and eigenvectors of $A(t)$ can be chosen real analytically in t, on the whole parameter domain.*

This result allows us to state the following general Lyapunov differential inequality, with time dependent coefficients:

Theorem 3.2. *Let $\beta(t)$ be a continuous curve in $\mathcal{L}(S_d)$, admissible for all $t\geq 0$. Suppose $X:\mathbb{R}_+\to S_d$, $t\mapsto X(t)$ is continuously differentiable. Then, the following assertions hold.*

(1) *If X is a solution of $\dot X(t)\succeq\beta(t)(X(t))$ for all $t\geq 0$ and $X(0)\succeq 0$, then $X(t)\succeq 0$, for all $t\geq 0$.*
(2) *If, in addition, $X(t)$ is analytic in $t\geq 0$ and solves $\dot X(t)\succeq\beta(t)(X(t))$ for all $t\geq 0$, then $X(0)\succ 0$ implies $X(t)\succ 0$, for all $t\geq 0$.*

Proof. For part (1), we infer the existence of a continuous curve $\gamma(t)\in S_d^+$ such that

$$\dot X(t)=\beta(t)(X(t))+\gamma(t),\quad X(0)\succeq 0.$$

Then the assertion is a consequence of Theorem 2.2.
So let us proceed with the proof of statement (2): According to Theorem 3.1, we may choose a C^1-curve[2] $\mathbb{R}_+\to O(d)$, $t\mapsto U(t)$ such that $Y(t):=U(t)X(t)U^T(t)$ is diagonal, for each $t\geq 0$. Then, we have

$$\begin{aligned}\dot Y(t)&=U(t)\dot X(t)U^T(t)+\dot U(t)X(t)U^T(t)+U(t)X(t)\dot U^T(t)\\&\succeq U(t)\beta(X(t))U^T(t)+\dot U(t)X(t)U^T(t)+U(t)X(t)\dot U^T(t)\\&=U(t)\beta(U^T(t)Y(t)U(t))U^T(t)+\dot U(t)U^T(t)Y(t)+Y(t)U(t)\dot U^T(t)\\&=\widetilde\beta(t)(Y(t)),\end{aligned}$$

[2] Actually, even an analytic curve may be chosen.

where for all $t \geq 0$, $\widetilde{\beta}(t) \in \mathcal{L}(S_d)$. Observe that $\widetilde{\beta}(t)$ is admissible for each $t \geq 0$. Indeed, if $u_0 \in \partial S_d^+$ and $v \in N_{S_d^+}(u_0)$, then $\langle U(t)^T u_0 U(t), U(t)^T v U(t)\rangle$=0 and we obtain by the cyclic property of the trace and the admissibility of $\beta(t)$

$$\begin{aligned}\langle \widetilde{\beta}(t)(u_0), v\rangle &= \langle U(t)\beta(U^T(t)u_0U(t))U^T(t), v\rangle + \langle \dot{U}(t)U^T(t)u_0, v\rangle + \langle u_0 U(t)\dot{U}^T(t), v\rangle \\ &= \langle \beta(U^T(t)u_0U(t)), U(t)^T vU(t)\rangle \geq 0.\end{aligned}$$

This, in particular, implies that $\widetilde{\beta}(t)_{ii}{}^{kk} \geq 0$ whenever $k \neq i$ (see Remark 2.6, (3)). Due to the diagonal form of $Y(t)$, $\dot{Y}(t)_{ii}$ $(i = 1, \ldots, d)$ satisfies

$$\dot{Y}_{ii}(t) \geq \sum_{k=1}^{d} \widetilde{\beta}(t)_{ii}{}^{kk} Y_{kk}(t), \quad Y_{ii}(0) > 0,$$

since $X(0) \succ 0$ implies $Y_{ii}(0) > 0$. By the non-negativity of $\widetilde{\beta}(t)_{ii}{}^{kk}$ $(i \neq k)$ it follows that

$$\dot{Y}_{ii}(t) \geq \widetilde{\beta}(t)_{ii}{}^{ii} Y_{ii}(t), \quad Y_{ii}(0) > 0.$$

Standard comparison implies that for $i = 1, \ldots, d$ and for all $t \geq 0$, we have

$$Y_{ii}(t) \geq e^{\int_0^t \widetilde{\beta}(s)_{ii}{}^{ii} ds} Y_{ii}(0) > 0.$$

As Y is diagonal, this is equivalent to $Y(t) \in S_d^{++}$. Hence $X(t) \in S_d^{++}$ too, which yields assertion (2). □

As a corollary, we have:

Corollary 3.3. *Let $X(t,u)$ solve for $t \geq 0$ the generalized Lyapunov differential equation,*

$$\dot{X}(t,u) = \beta(t)(X(t,u)) + \gamma(t), \quad X(0,u) = u \in S_d, \tag{3.1}$$

with continuous functions $\beta(t), \gamma(t)$, admissible for all $t \geq 0$. Then, we have:

(1) *If $u \succeq 0$, then $X(t,u) \succeq 0$, for all $t \geq 0$.*
(2) *In addition, assume that $\beta(t)$ and $\gamma(t)$ are analytic in $t \geq 0$. Then $u \succ 0$ implies $X(t,u) \succ 0$, for all $t \geq 0$.*

Proof. The first assertion is an immediate consequence of part (1) of the preceding theorem, whereas the second one holds in view of the analyticity of $X(t)$ and part (2) of Theorem 3.2. □

From the above corollary, it follows directly that for two solutions X_1 and X_2 of (3.1), $X_1(0) \preceq X_2(0)$ implies $X_1(t) \preceq X_2(t)$ for all $t \geq 0$, since $X_2(0) - X_1(0) \succeq 0$ and $\dot{X}_2(t) - \dot{X}_1(t) = \beta(t)(X_2(t) - X_1(t))$. Clearly, with the additional assumption of analyticity of $\beta(t)$, $\preceq$ can be replaced by $\prec$. Hence, we have an *order-preserving property* for all $t \geq 0$, however not for $t < 0$ as illustrated in the example below.

Example 3.3.1. Consider for $d = 2$, the linear admissible map $u \mapsto \beta(u) = \operatorname{diag}(u_{22}, u_{11})$ and the corresponding Lyapunov differential equation:

$$\dot{X}(t,u) = \beta(X(t,u)) = \operatorname{diag}(X_{22}(t,u), X_{11}(t,u)), \quad X(0,u) = u \in S_d.$$

Its unique global solution on $\mathbb{R}$ is given by

$$X(t,u) = \begin{pmatrix} u_{11}\cosh(t) + u_{22}\sinh(t) & u_{12} \\ u_{12} & u_{11}\sinh(t) + u_{22}\cosh(t) \end{pmatrix}.$$

For times $t \geq 0$, order is obviously preserved, whereas for $t < 0$ this does not hold true. For example, let $u, v \in S_2^+$ with $u_{11} = v_{11} = u_{12} = v_{12} = 0$, and $u_{22} < v_{22}$. Then for all $t > 0$,

$$X_{11}(-t,u) = -u_{22}\sinh(t) > -v_{22}\sinh(t) = X_{11}(-t,v).$$

4. Generalized matrix Riccati differential equations

As an application of the results of the preceding section, we prove a comparison theorem for generalized matrix Riccati differential equations of the form

$$\dot{X}(t,u) = -X(t,u)\alpha(t)X(t,u) + \beta(t)(X(t,u)) + \gamma(t), \quad X(0,u) = u \in S_d, \tag{4.1}$$

with admissible parameters α, β, γ. By means of these results, we shall get global existence for the semi-flow $\mathbb{R}_+ \times S_d^+ \ni (t,u) \mapsto X(t,u)$.

As standard matrix Riccati theory is not exactly applicable here, we elaborate suitable comparison arguments by mimicking those from the theory of classical matrix Riccati differential equations (cf. [1], chapter 4). Our statements strongly rely on the order preserving properties of the Lyapunov equation as established in Theorem 3.2.

Definition 4.1. For two admissible parameter sets $(\alpha_i, \beta_i, \gamma_i)$ $(i = 1, 2)$, we define the order $\leq_T$ as follows: $(\alpha_1, \beta_1, \gamma_1) \leq_T (\alpha_2, \beta_2, \gamma_2)$ if and only if

$$\alpha_2 \preceq \alpha_1, \quad (\forall u \in S_d^+)(\beta_1(u) \preceq \beta_2(u)), \quad \gamma_1 \preceq \gamma_2.$$

If $(\alpha_i, \beta_i, \gamma_i)$ $(i = 1, 2)$ are time-dependent admissible parameters, on an interval $I \subset \mathbb{R}$, then we write, by an abuse of notation, $(\alpha_1, \beta_1, \gamma_1) \leq_T (\alpha_2, \beta_2, \gamma_2)$ if and only if the relation is satisfied for each $t \in I$.

We observe that $\leq_T$ is a partial order.

Theorem 4.2. *(Comparison Theorem) Let $X_i(t)$, $i = 1, 2$ be solutions of*

$$\dot{X}_i(t) = -X_i(t)\alpha_i(t)X_i(t) + \beta_i(t)(X_i(t)) + \gamma_i(t) \tag{4.2}$$

on $I = [0, T)$ with continuous admissible parameters satisfying

$$(\alpha_1, \beta_1, \gamma_1) \leq_T (\alpha_2, \beta_2, \gamma_2).$$

Then, $0 \preceq X_1(0) \preceq X_2(0)$ implies $X_1(t) \preceq X_2(t)$ for all $t \in I$. If, in addition, $X_i(t)$ are analytic for $i = 1, 2$, then $0 \preceq X_1(0) \prec X_2(0)$ implies $X_1(t) \prec X_2(t)$ for all $t \in I$.

Proof. We set $X(t) := X_2(t) - X_1(t)$ and $\beta_0(t) := -\frac{1}{2}X(t)\alpha_2(t) - X_1(t)\alpha_2(t) \in M_d(\mathbb{R})$ and define the (curve of) linear maps $\tilde{\beta}(t)$ by $u \mapsto \tilde{\beta}(t)(u) := \beta_0(t)u + u\beta_0(t)^T + \beta_2(t)(u)$. By Remark 2.6, 2, $\tilde{\beta}(t)$ is admissible as well. Moreover, $(\alpha_1, \beta_1, \gamma_1) \leq_T (\alpha_2, \beta_2, \gamma_2)$ implies

$$(\forall u \in S_d^+)(\forall t \geq 0)\left(u(\alpha_1(t) - \alpha_2(t))u + (\beta_2(t) - \beta_1(t))(u) + (\gamma_2(t) - \gamma_1(t)) \succeq 0\right). \tag{4.3}$$

Therefore, X satisfies the following differential inequality (where we suppress time-dependence throughout),

$$\begin{aligned}
\dot{X} &= \dot{X}_2 - \dot{X}_1 = -X_2\alpha_2X_2 + \beta_2(X_2) + \gamma_2 - (-X_1\alpha_1X_1 + \beta_1(X_1) + \gamma_1) \\
&= -(X_2 - X_1)\alpha_2(X_2 - X_1) - X_1\alpha_2(X_2 - X_1) - (X_2 - X_1)\alpha_2X_1 \\
&\quad + \beta_2(X_2 - X_1) + [X_1(\alpha_1 - \alpha_2)X_1 + (\beta_2 - \beta_1)(X_1) + (\gamma_2 - \gamma_1)] \\
&= \beta_0(t)(X_2 - X_1) + (X_2 - X_1)\beta_0(t)^T + \beta_2(X_2 - X_1) \\
&\quad + [X_1(\alpha_1 - \alpha_2)X_1 + (\beta_2 - \beta_1)(X_1) + (\gamma_2 - \gamma_1)] \\
&\succeq \tilde{\beta}(X).
\end{aligned}$$

Here the last inequality is a consequence of equation (4.3) and the fact that $X_1 \succeq 0$ according to the invariance principle of section 2. Hence, by Theorem 3.2 we have $X(t) \succeq 0$ (resp. $X(t) \succ 0$), for all $t \in I$. □

Theorem 4.3. *(Global existence of semi-flows) For each continuous admissible parameter set $t \mapsto (\alpha, \beta, \gamma)(t)$, the semi-flow $X(t) = X(t,u)$ corresponding to the solution of*

$$\dot{X}(t) = -X(t)\,\alpha(t)\,X(t) + \beta(t)(X(t)) + \gamma(t), \quad X(0) = u \in S_d^+, \tag{4.4}$$

exists for all times $t \geq 0$ and $X(t,u) \in S_d^+$. Furthermore, if $t \mapsto (\alpha, \beta, \gamma)(t)$ is analytic in $t \geq 0$ and $u \in S_d^{++}$, then $X(t,u) \in S_d^{++}$, for all times $t \geq 0$.

Proof. Denote by I the maximal interval of existence of $X(t)$. Clearly, there exists some $T>0$ such that $[0,T)\in I$. First, we compare $X(t)$ with $Z(t)$, the solution of the generalized inhomogeneous Lyapunov equation,

$$\dot{Z}(t)=\beta(t)(Z(t))+\gamma(t),\quad Z(0)=u\in S_d^+, \tag{4.5}$$

which admits global solutions, as it is a linear ODE. Since for all $t\geq 0$, $(\alpha(t),\beta(t),\gamma(t))\leq_T (0,\beta(t),\gamma(t))$, we have by Theorem 4.2, $X(t)\preceq Z(t)$ for all $t\in I$. To find a lower bound for $X(t)$, we simply compare it with the trivial solution $Y(t)=0$ of the initial value problem

$$\dot{Y}(t)=-Y(t)\alpha(t)Y(t)+\beta(t)(Y(t)),\qquad Y(0)=0.$$

In this situation, $(\alpha,\beta,0)\leq_T(\alpha,\beta,\gamma)$ and for the initial data we have $0=Y(0)\preceq X(0)=u$. Hence, by Theorem 4.2 we obtain $0\preceq X(t)$ for all $t\in I$ (resp. $0\prec X(t)$ by the analyticity of $X(t)$). Therefore $X(t)$ cannot blow up in finite positive time, that is, $I=\mathbb{R}_+$. □

5. Related Problems and Conclusion

In this paper, we have elaborated comparison results for a generalization of symmetric matrix Riccati differential equations. Our results concerning the strict order $\prec$ induced by the cone S_d^{++} are sharper than the ones available in the literature (see [7, 8] and the references therein). Indeed, Volkmann [8] develops comparison results in the abstract setting of an infinite-dimensional Banach-space E ordered by a cone K with non-empty interior (with respective order relations $\preceq$ and $\prec$). His results are generalizations of Walter's (concerning the cone $\mathbb{R}_+^d$) and of others. With respect to $\preceq$, his results are comparable with ours. However, what concerns $\prec$, Volkmann's statement asserts that $X(t)\prec Y(t)$ holds for all times t of existence under the assumption of

$$\dot{X}(t)-R(t,X(t))\prec\dot{Y}(t)-R(t,Y(t)),$$

and $X(0)\prec Y(0)$, where $R(t,\cdot)$ is a quasi-monotonic map (this notion is equivalent to ours of admissibility). Hence, our theorems and corollaries improve Volkmann's in the present finite-dimensional setting. Moreover, we conjecture that all statements hold even without the assumption of analyticity (thus fully generalizing the respective results concerning the classical matrix Lyapunov and Riccati differential equation). It should be noted, however, that our method is tailored to the analytic setting, and therefore it is not applicable in a more general context. For counterexamples concerning smooth choices of eigenvectors of non-analytic curves $X(t)$, we refer to [2].

The method applied in the present work can be best described as a diagonalization procedure, which transfers the problem of ODE comparison on the cone S_d^+ into a problem on the cone $\mathbb{R}_+^d$, $d\geq 2$. Our main interest in the sharp inequality $X(t,u)\succ 0$ for $X(0)=u\succ 0$ arises from our work on affine processes on S_d^+. Indeed, the fact that the function $\psi(t,u)$ as introduced in (1.3) can never reach the boundary if the initial value u lies in the interior, allows us to prove the Feller property of affine processes on S_d^+. One necessary step is to show that for each $u\in S_d^{++}$ and $t\geq 0$, the Laplace transform (1.1) is a continuous function in x, vanishing at infinity. Therefore we need that $\psi(t,u)\in S_d^{++}$ for all $t\geq 0$.

Ongoing research elaborates on extending the mentioned diagonalization procedure to allow a treatment of Riccati differential equations $\dot{X}=R(t,X)$ with a right hand side $R(t,\cdot)$ which is analytic in the interior of the cone, however, fails to be Lipschitz on the boundary. Such functions naturally arise as cumulant-generating functions of infinite-divisible random variables on S_d, which are connected to the class of affine jump-diffusion processes on S_d^+.

References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati equations*, Systems & Control: Foundations & Applications, Birkhäuser Verlag, Basel, 2003. In control and systems theory.

[2] D. Alekseevsky, A. Kriegl, P. W. Michor, and M. Losik, *Choosing roots of polynomials smoothly*, Israel J. Math., 105 (1998), pp. 203–233.

[3] C. Cuchiero, D. Filipovic, E. Mayerhofer, and J. Teichmann, *Affine processes on the cone of non-negative definite symmetric matrices*. 2008.

[4] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[5] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*, vol. 317 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1998.
[6] A. N. Stokes, *A special property of the matrix Riccati equation*, Bull. Austral. Math. Soc., 10 (1974), pp. 245–253.
[7] R. Uhl, *Ordinary differential inequalities and quasimonotonicity in ordered topological vector spaces*, Proc. Amer. Math. Soc., 126 (1998), pp. 1999–2003.
[8] P. Volkmann, *Gewöhnliche Differentialungleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen*, Math. Z., 127 (1972), pp. 157–164.
[9] W. Walter, *Gewöhnliche Differentialgleichungen*, Springer-Lehrbuch. [Springer Textbook], Springer-Verlag, Berlin, fifth ed., 1993. Eine Einführung. [An introduction].

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